(* Example of proof technique for GoldenRatio *)

$$\phi^n == \phi^n$$

(* For n>1, *)

$$-\left(-\frac{1}{\phi}\right)^n + \left(\phi^n + \left(-\frac{1}{\phi}\right)^n\right) + \text{Round}\left[-\left(-\frac{1}{\phi}\right)^n\right] == \phi^n$$

(* ϕ and $\left(-\frac{1}{\phi}\right)$ are the roots of x^2-x-
1. Since this is monic and has integer coefficients,
and since $\phi-\frac{1}{\phi}$ is an integer, *)

$$-\left(-\frac{1}{\phi}\right)^n + \text{Round}\left[\phi^n + \left(-\frac{1}{\phi}\right)^n - \left(-\frac{1}{\phi}\right)^n\right] == \phi^n$$

$$-\left(-\frac{1}{\phi}\right)^n + \text{Round}[\phi^n] == \phi^n$$

$$\text{Round}[\phi^n] - \phi^n == \left(-\frac{1}{\phi}\right)^n$$

(* and summing over
$n\in[1,\infty]$ while respecting that the above conclusions are only valid for n>1, *)

$$\sum_{n=1}^{\infty}(\text{Round}[\phi^n] - \phi^n) == 2 - \phi + \sum_{n=2}^{\infty}\left(-\frac{1}{\phi}\right)^n$$

$$\sum_{n=1}^{\infty}(\text{Round}[\phi^n] - \phi^n) == \frac{1}{\phi}$$

(* The full version: let α be a Pisot-Vijayaraghavan number,
and r_i, $1\le i\le d$ be the Galois
conjugates. (d is the degree of the minimum polynomial, minus 1.)
Furthermore, let k be an integer such that
$\forall_{n\ge k}\text{Abs}\left[\sum_{i=1}^{d} r_i^n\right] < \frac{1}{2}$ – e.g. such that $\sum_{i=1}^{d}\text{Abs}\left[r_i^k\right] < \frac{1}{2}$. Then, for $n\ge k$,
*)

$$-\sum_{i=1}^{d} r_i^n + \alpha^n + \sum_{i=1}^{d} r_i^n + \text{Round}\left[-\sum_{i=1}^{d} r_i^n\right] == \alpha^n$$

(* Since α must be a root of an irreducible monic polynomial
of degree d with integer coefficients, and the sum of α along
with its Galois conjugates is an integer by Vieta's formulas,
*)

$$-\sum_{i=1}^{d} r_i^n + \text{Round}\left[\alpha^n + \sum_{i=1}^{d} r_i^n - \sum_{i=1}^{d} r_i^n\right] == \alpha^n$$

$$\text{Round}[\alpha^n] - \alpha^n == \sum_{i=1}^{d} r_i^n \tag{1}$$

(* Now, summing over n∈[k,∞] *)

$$\sum_{n=k}^{\infty} (\text{Round}[\alpha^n] - \alpha^n) == \sum_{n=k}^{\infty} \sum_{i=1}^{d} r_i^n$$

$$\sum_{n=k}^{\infty} (\text{Round}[\alpha^n] - \alpha^n) == \sum_{i=1}^{d} \sum_{n=k}^{\infty} r_i^n$$

$$\sum_{n=0}^{\infty} \left(\text{Round}[\alpha^n] - \alpha^n\right) == \sum_{n=0}^{k-1} \left(\text{Round}[\alpha^n] - \alpha^n\right) + \sum_{i=1}^{d} \frac{r_i{}^k}{1 - r_i} \tag{2}$$

(* We can also raise both sides of (1) to the p_1th power, getting *)

$$(\text{Round}[\alpha^n] - \alpha^n)^{p_1} == \left(\sum_{i=1}^{d} r_i^n\right)^{p_1}$$

(* And finally, because Bill wanted me to spell it out,
here it is with an extra q^k on the denominator: *)

$$\sum_{n=k}^{\infty} \frac{\prod_{j=1}^{j=na} (\text{Round}[\alpha^n] - \alpha^n)}{q^n} == \sum_{n=k}^{\infty} \frac{\prod_{j=1}^{j=na} \sum_{i=1}^{d_j} r_i^n}{q^n}$$

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{j=na} \left(\text{Round}[\alpha_j^n] - \alpha_j^n\right)}{q^n} == \sum_{n=0}^{k-1} \frac{\prod_{j=1}^{j=na} \left(\text{Round}[\alpha_j^n] - \alpha_j^n\right)}{q^n} + \sum_{n=k}^{\infty} \frac{\prod_{j=1}^{j=na} \sum_{i=1}^{d_j} r_{j,i}^n}{q^n}$$

(* Simplifying the RHS
(first-time readers should skip this part. No, really. I mean it.) *)

$$\sum_{n=k}^{\infty} \frac{\prod_{j=1}^{j=na} \sum_{i=1}^{d_j} r_{j,i}^n}{q^n}$$

$$\sum_{n=k}^{\infty} \frac{\sum_{1 \le i_1 \le d_1, \ldots, 1 \le i_{na} \le d_{na}} \prod_{j=1}^{na} r_{j,i_j}^n}{q^n}$$

(* above this line is okay *)

(* lines below this are not *)

$$\sum_{n=k}^{\infty} \sum_{1 \le i_1 \le d_1, \ldots, 1 \le i_{na} \le d_{na}} \frac{\prod_{j=1}^{na} r_{j,i_j}^n}{q^n}$$

$$\sum_{1 \le i_1 \le d_1, \ldots, 1 \le i_{na} \le d_{na}} \sum_{n=k}^{\infty} \frac{\prod_{j=1}^{na} r_{j,i_j}^n}{q^n}$$

$$\sum_{1 \le i_1 \le d_1, \ldots, 1 \le i_{na} \le d_{na}} \sum_{n=k}^{\infty} \frac{\left(\prod_{j=1}^{na} r_{j,i_j}\right)^n}{q^n}$$

$$\sum_{1\le i_1\le d_1,\,\ldots,\,1\le i_{na}\le d_{na}} \sum_{n=k}^{\infty} \left(\frac{\left(\prod_{j=1}^{na} r_{j,i_j}\right)}{q}\right)^n$$

$$\sum_{1\le i_1\le d_1,\,\ldots,\,1\le i_{na}\le d_{na}} \frac{\left(\frac{\prod_{j=1}^{na} r_{j,i_j}}{q}\right)^k}{\left(1-\frac{\prod_{j=1}^{na} r_{j,i_j}}{q}\right)}$$

$$\sum_{1\le i_1\le d_1,\,\ldots,\,1\le i_{na}\le d_{na}} \frac{\prod_{j=1}^{na} r_{j,i_j}^k}{q^k\left(1-\frac{\prod_{j=1}^{na} r_{j,i_j}}{q}\right)}$$

$$\sum_{1\le i_1\le d_1,\,\ldots,\,1\le i_{na}\le d_{na}} \frac{\prod_{j=1}^{na}\left(r_{j,i_j}^k\right)}{q^k - q^{k-1}\prod_{j=1}^{na} r_{j,i_j}} \qquad (3)$$

```
(* Which seems to be about as simple as it gets. *)

(* Finally, we have *)
```

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{j=na}\left(\mathrm{Round}\left[\alpha_j^n\right]-\alpha_j^n\right)}{q^n} == \sum_{n=0}^{k-1} \frac{\prod_{j=1}^{j=na}\left(\mathrm{Round}\left[\alpha_j^n\right]-\alpha_j^n\right)}{q^n} + \sum_{1\le i_1\le d_1,\,\ldots,\,1\le i_{na}\le d_{na}} \frac{\prod_{j=1}^{na}\left(r_{j,i_j}^k\right)}{q^k - q^{k-1}\prod_{j=1}^{na} r_{j,i_j}}$$

```
(* FUNCTIONS TO DO THE SUMMATION: *)

pisotNumberQ[a_] := AlgebraicIntegerQ[a] && Element[a, Reals] && (a > 1) &&
  (Count[List @@ (Last /@ Roots[MinimalPolynomial[a, x] == 0, x]), _?(Abs[#] > 1 &)] == 1)

roundoffsum[a_] :=
 If[(! pisotNumberQ[a]) || (InexactNumberQ[a]),
  Print["roundoffsum::nonpisot - This isn't a Pisot number!"],
  Block[{minpoly = MinimalPolynomial[a], conjugates, k},
   (conjugates = Select[#[[1, 2]] & /@ Solve[minpoly[x] == 0, x], Abs[#] < 1 &];
    (*Compute k satisfying Sum_{i=0}^{d-1} Abs[r_i^k] < 1/2 *)
    k = getRoundoffCutoff[conjugates];
    Print["minpoly degree " <>
      ToString[Length[conjugates] + 1] <> ", k=" <> ToString[k]];
    Sum_{n=0}^{k-1} (Round[a^n] - a^n) + Sum_{i=1}^{Length[conjugates]} conjugates[[i]]^k / (1 - conjugates[[i]])
   )]]

getConjugates[a_] :=
 Select[#[[1, 2]] & /@ Solve[MinimalPolynomial[a, x] == 0, x], Abs[#] < 1 &]

getRoundoffCutoff[conjugates_] := Block[{k = 0},
  (While[Sum_{i=1}^{Length[conjugates]} Abs[conjugates[[i]]]^k >= 1 / 2, k++]; k)]
```

```
roundoffsum[a_List, q_] := Block[{ks, k, conjugates, ft, rd, st, prod, iter},
  (conjugates = getConjugates /@ a;
   ks = getRoundoffCutoff /@ conjugates;
   k = Max[ks];
   Print["ks are " ~~ ToString[ks]];
   rd = Times @@ (Round[#^n] - #^n & /@ a);
   ft = Sum[rd/q^n, {n, 1, k-1}];
   iter = Sequence @@ Table[{i_j, 1, Length[conjugates[[j]]]}, {j, 1, Length[a]}];
   Print[iter];
   st = 1/q^(k-1) Total[Flatten[
       Table[(prod = Product[conjugates[[j, i_j]], {j, 1, Length[a]}];
          prod^k/(q - prod)), Evaluate[iter]]
       ]
     ];
   Return[ft + st];
  )]
```