

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\text{Matrix proof that } 1^{-2} + 2^{-2} + 3^{-2} + \dots = \frac{\pi^2}{6}$$

Abstract

Representing sums by matrix products, we transform the $\zeta(2)$ series into the series for $6 \sin^{-1}(\frac{1}{2})^2$. Additional examples illustrate the power of this technique.

Path invariance

Matrix products subsume sums

2x2 matrix multiplication is especially simple when the bottom rows are (0,1):

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \cdot c & a \cdot d + b \\ 0 & 1 \end{pmatrix},$$

yielding another 2x2 of the same sort. We can use iterated products of these as an alternative to summation notation:

$$\prod_{n=1}^k \begin{pmatrix} r_n & a_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_1 r_2 \dots r_k & a_1 + r_1(a_2 + r_2(\dots + r_{k-1} a_k) \dots) \\ 0 & 1 \end{pmatrix}.$$

Thus we could compute six terms of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with

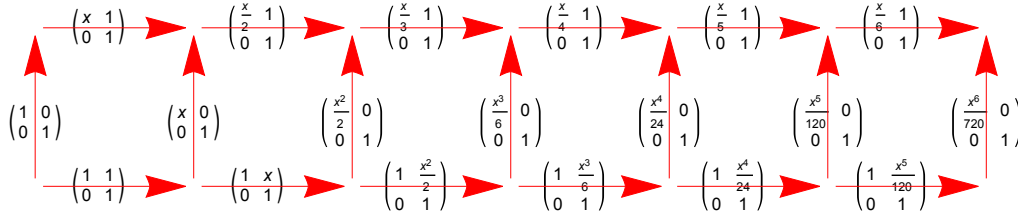
$$\prod_{n=0}^5 \begin{pmatrix} 1 & \frac{x^n}{n!} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & \frac{x^5}{120} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \\ 0 & 1 \end{pmatrix}.$$

This is equivalent to $\sum$ notation, plus some extra work. But we could actually *save* some work by multiplying instead the following six much simpler matrices:

$$\prod_{n=1}^6 \begin{pmatrix} \frac{x}{n} & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{x}{2} & 1 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} \frac{x}{6} & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x^6}{720} & \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \\ 0 & 1 \end{pmatrix}.$$

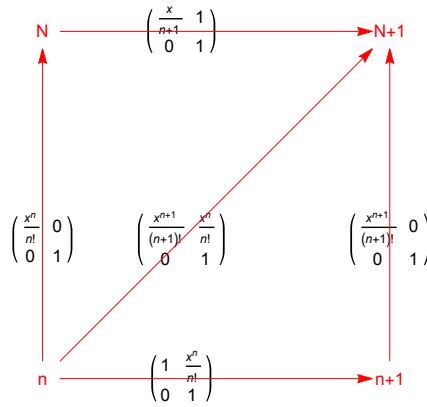
We got our desired sum in the UR (upper right), and as a bonus, the first neglected term (instead of 1) in the UL.

We can eliminate those factorials and n th powers from the n th matrix with the help of a path invariance diagram:



We compute our original (extra work) product by following the six arrows across the bottom, multiplying their associated matrices as we go. If we then multiply by the matrix on the rightmost vertical arrow, we simply change the UL 1 to $\frac{x^6}{720}$, producing what we got from the "factorial free" product corresponding to the top row of this diagram. Tossing in for good measure that identity matrix on the leftmost vertical arrow, both paths then start at the lower left and end at the upper right with the same product. But where did we get those matrices in the top row? They were chosen to make every arrow-product path-invariant, *i.e.*, dependent only on the endpoints. We are even allowed to traverse an arrow backwards, provided we invert its matrix before multiplying.

It suffices to impose path invariance on every little square:



You can readily verify that both "taxicab" paths from n to $N+1$ produce the matrix on the diagonal "shortcut". Thus the matrix on the top arrow is the product of the inverse of the left-hand arrow, the bottom arrow, and the right-hand arrow:

$$\begin{pmatrix} \frac{x^n}{n!} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & \frac{x^n}{n!} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{x^{n+1}}{(n+1)!} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x}{n+1} & 1 \\ 0 & 1 \end{pmatrix},$$

$$N \rightarrow n \rightarrow n+1 \rightarrow N+1 = N \rightarrow N+1.$$

But then where do we get the matrices on the vertical arrows? They are simply diagonal matrices whose UL is whatever we want to divide out of the UR of the matrix we are simplifying. In this case we divided out the whole series term, changing the matrix from "term form" to "term-ratio form".

Let us call diagrams of this $2 \times n$ shape "two railers". With more general "cross-tie" matrices (with nonzero UR), it is possible to convert any sum of k terms to any other, with the discrepancy all piled up in the rightmost cross-tie. Thus, to convert between infinite sums, that UR element must $\rightarrow 0$ as $k \rightarrow \infty$, or at worst approach a known, finite quantity.

Besides more general matrices, we can have more general diagrams. Here is an unusually symmetric

four-dimensional path-invariant system of 2×2 s, whose diagram is infinite in all directions:

$$I_{i,j,k,n} = \begin{pmatrix} \frac{(i+k)(i+n)}{i(i+j+k+n)} & -\frac{1}{i} \\ 0 & 1 \end{pmatrix},$$

$$J_{i,j,k,n} = \begin{pmatrix} \frac{(j+k)(j+n)}{j(i+j+k+n)} & -\frac{1}{j} \\ 0 & 1 \end{pmatrix},$$

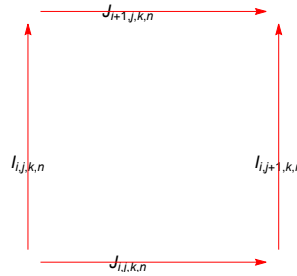
$$K_{i,j,k,n} = \begin{pmatrix} \frac{(i+k)(j+k)}{k(i+j+k+n)} & \frac{1}{k} \\ 0 & 1 \end{pmatrix},$$

$$N_{i,j,k,n} = \begin{pmatrix} \frac{(i+n)(j+n)}{n(i+j+k+n)} & \frac{1}{n} \\ 0 & 1 \end{pmatrix}.$$

Each of these four matrix-valued functions labels the arrows in one of the four coordinate directions of the grid. All four arrows departing from a given node have the same four subscripts, which provide a natural label for that node. An illustrative closed path starting and ending at the node (a,b,c,d) (which need not be integers, nor even real) would generate the product

$$J_{a,b,c,d} I_{a,b+1,c,d} N_{a+1,b+1,c,d} K_{a+1,b+1,c,d+1} I_{a,b+1,c+1,d+1}^{-1} N_{a,b+1,c+1,d}^{-1} K_{a,b+1,c,d}^{-1} J_{a,b,c,d}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

because J 's arrow points in the second coordinate direction, I 's in the first, etc. Path invariance guarantees that the closed loop generates the identity matrix. To demonstrate path invariance in this grid, it is sufficient to check



and similarly for (J,K) and (K,N) .

We shall use this system to prove things about $\zeta(2)$, and the functions arctan and arcsin.

Two more series for $\zeta(2,a)$

As a warmup, consider the two taxicab paths for the (integer-sided) rectangle with opposite corners $(0,0,1,a)$ (for various a) and $(0,0,\kappa,u)$, and then let $\kappa, u \rightarrow \infty$. We will need to discard the I and J matrices, which divide by 0 when $i=j=0$. This leaves the 2-dimensional system

$$K_{k,n} = \begin{pmatrix} \frac{(i+k)(j+k)}{(i+j+k+n)k} & \frac{1}{k} \\ 0 & 1 \end{pmatrix},$$

$$N_{k,n} = \begin{pmatrix} \frac{(i+n)(j+n)}{(i+j+k+n)n} & \frac{1}{n} \\ 0 & 1 \end{pmatrix},$$

where i and j are now arbitrary constants. Setting them to 0 leaves merely

$$K_{k,n} = \begin{pmatrix} \frac{k}{k+n} & \frac{1}{k} \\ 0 & 1 \end{pmatrix},$$

$$N_{k,n} = \begin{pmatrix} \frac{n}{n+k} & \frac{1}{n} \\ 0 & 1 \end{pmatrix}.$$

It is good practice to keep a handy "check path invariance" function to invoke whenever your matrices change appreciably. In this case, it finds that we have not blundered, and indeed

$$N_{k,n} K_{k,n+1} = K_{k,n} N_{k+1,n} = \begin{pmatrix} \frac{k n}{(k+n)(k+n+1)} & \frac{k^2+k n+n^2}{k n (k+n)} \\ 0 & 1 \end{pmatrix},$$

i.e., both taxicab paths amount to the same diagonal shortcut for the typical unit square. (The infinite sum generated by chaining together these shortcut matrices will later prove very interesting and valuable to us.) Our rectangle is now $(1, a) \leq (k, n) < (\kappa, \upsilon)$. The $a \leq n < \upsilon$ leg (with $k=1$, and $\upsilon-a$ an integer) is

$$\prod_{n=a}^{\upsilon-1} N_{1,n} = \prod_{n=n_0}^{\upsilon-1} \begin{pmatrix} \frac{n}{n+1} & \frac{1}{n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{n_0}{\upsilon} & \sum_{n=a}^{\upsilon-1} \frac{n_0}{n^2} \\ 0 & 1 \end{pmatrix}.$$

If this last step was unclear, a cross-tie of

$$\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

will convert the product back to term form:

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \prod_{n=a}^{\upsilon-1} \begin{pmatrix} 1 & \frac{1}{n^2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \sum_{n=a}^{\upsilon-1} \frac{a}{n^2} \\ 0 & 1 \end{pmatrix}.$$

The $1 \leq k < \kappa$ leg (with $n=\upsilon$) is

$$\prod_{k=1}^{\kappa-1} K_{k,\upsilon} = \prod_{k=1}^{\kappa-1} \begin{pmatrix} \frac{k}{k+\upsilon} & \frac{1}{k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} O(\upsilon^{-\kappa}) & O(\upsilon^0) \\ 0 & 1 \end{pmatrix}.$$

We care only that this is finite. The $1 \leq k < \kappa$ leg (with $n=a$) is

$$\prod_{k=1}^{\kappa-1} K_{k,a} = \prod_{k=1}^{\kappa-1} \begin{pmatrix} \frac{k}{k+a} & \frac{1}{k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \prod_{k=1}^{\kappa-1} \frac{k}{k+a} & \sum_{k=1}^{\kappa-1} \frac{\prod_{j=1}^{k-1} \frac{j}{j+a}}{k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\binom{\kappa+a-1}{\kappa-1}} & \sum_{k=1}^{\kappa-1} \frac{1}{k \binom{k+a-1}{k-1}} \\ 0 & 1 \end{pmatrix}.$$

Finally, the $a \leq n \leq \upsilon$ leg (with $k=\kappa$)

$$\prod_{n=a}^{\upsilon-1} N_{\kappa,n} = \prod_{n=a}^{\upsilon-1} \begin{pmatrix} \frac{n}{n+\kappa} & \frac{1}{n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} O(\kappa^{-\upsilon}) & O(\kappa^0) \\ 0 & 1 \end{pmatrix}.$$

As with the $\prod K_{k,\upsilon}$ leg, we care only that this is finite. Path equivalence for the entire rectangle is now

$$\prod_{n=a}^{\upsilon-1} N_{1,n} \prod_{k=1}^{\kappa-1} K_{k,\upsilon} = \prod_{k=1}^{\kappa-1} K_{k,a} \prod_{n=a}^{\upsilon-1} N_{\kappa,n},$$

or

$$\begin{pmatrix} \frac{a}{\upsilon} & \sum_{n=a}^{\upsilon-1} \frac{a}{n^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O(\upsilon^{-\kappa}) & O(\upsilon^0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\binom{\kappa+a-1}{\kappa-1}} & \sum_{k=1}^{\kappa-1} \frac{1}{k \binom{k+a-1}{k-1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} O(\kappa^{-\upsilon}) & O(\kappa^0) \\ 0 & 1 \end{pmatrix}.$$

Letting κ and $\nu \rightarrow \infty$ (independently) while assuming, for convergence, $a > 0$,

$$\begin{pmatrix} 0 & \sum_{n=a}^{\infty} \frac{n_a}{n^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & O(\nu^0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sum_{k=1}^{\infty} \frac{1}{k \binom{k+a-1}{k-1}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & O(\kappa^0) \\ 0 & 1 \end{pmatrix}.$$

Finally, equating UR elements,

$$\sum_{n \geq a} \frac{a}{n^2} = a \cdot \zeta(2, a) = \sum_{k \geq 1} \frac{1}{k \binom{k+a-1}{k-1}},$$

where $\zeta(s, a) := \sum_{n=a}^{\infty} n^{-s}$ is the Hurwitz zeta function, or in the special case $a=1$, the Riemann zeta function $\zeta(s)$. So for $a=1$, we have derived the tautology $\zeta(2) = \zeta(2)$. But for $a=2$, we get the slightly more interesting

$$2(\zeta(2) - 1) = \sum_{k \geq 1} \frac{2}{k^2(1+k)}.$$

For $a=1/2$

$$\frac{\zeta(2, 1/2)}{2} = \frac{3}{2} \cdot \zeta(2) = \sum_{k \geq 1} \frac{1}{k \binom{k-1/2}{k-1}},$$

the first equality following easily from

$$\zeta(s, 1/2) = (2^s - 1) \zeta(s),$$

i.e., 2^s times the odd terms of $\sum_{n \geq 1} n^{-s}$.

We should also mention the closely related Trigamma Function ψ_1 ,

$$\zeta(2, a) = \psi_1(a) := \frac{\partial^2 \log(\Gamma(a))}{\partial a^2}.$$

With $a=1/4$ we get Catalan's still mysterious constant

$$C := 1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + \dots,$$

$$\frac{\zeta(2, 1/4)}{4} = 2 \cdot C + \frac{3}{2} \cdot \zeta(2) = \sum_{k \geq 1} \frac{1}{k \binom{k-3/4}{k-1}},$$

or with $a=3/4$,

$$\frac{3 \cdot \zeta(2, 3/4)}{4} = \frac{9}{2} \cdot \zeta(2) - 6 \cdot C = \sum_{k \geq 1} \frac{1}{k \binom{k-1/4}{k-1}}.$$

Path-invariant matrices aren't always indifferent to how κ and ν approach infinity, but in this simple case, we can carelessly say that all paths from $(1, a)$ to (∞, ∞) yield $a \zeta(2, a)$, including the shortcut directly up the diagonal, formed from the matrices

$$\prod_{j \geq 0} N_{j+1, j+a} K_{j+1, j+a+1} = \prod_{j \geq 0} K_{j+1, j+a} N_{j+2, j+a} = \prod_{j \geq 0} \begin{pmatrix} \frac{(j+1)(j+a)}{(2j+a+1)(2j+a+2)} & \frac{3j^2+3aj+3j+a^2+a+1}{(j+1)(j+a)(2j+a+1)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \cdot \zeta(2, a) \\ 0 & 1 \end{pmatrix}.$$

We note that this shortcut matrix product converges *dramatically* faster than the sums taken along the edges—two bits per term versus zero, and that converting it to $\sum$ notation would be dramatically

messier. But why bother? Matrices are both easier to canonicalize for recognition, and more efficient for computation. Especially when we specialize to our case of interest, $a=1$:

$$\prod_{j \geq 1} N_{j,j} K_{j,j+1} = \prod_{j \geq 1} K_{j,j} N_{j+1,j} = \prod_{j \geq 1} \begin{pmatrix} \frac{j}{2(2j+1)} & \frac{3}{2j} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta(2) \\ 0 & 1 \end{pmatrix}.$$

E.g., taking 9 terms:

$$\prod_{j=1}^9 \begin{pmatrix} \frac{j}{2(2j+1)} & \frac{3}{2j} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{923780} & \frac{16929464521}{10291881600} \\ 0 & 1 \end{pmatrix},$$

the first neglected term being $\frac{3}{2 \times 10} \frac{1}{923780} \approx 4.87 \times 10^{-7}$. We shall now show that this exact same matrix product computes

$$\begin{pmatrix} 0 & 6 \sin^{-1}(1/2)^2 \\ 0 & 1 \end{pmatrix}.$$

The power series for $\sin^{-1}(y)^2$

The power series for $\tan^{-1}(x)$

$$\tan^{-1}(x) = \int_0^{\tan^{-1}(x)} 1 \cdot ds = \int_{s=0}^{\tan^{-1}(x)} \frac{\sec^2(s)}{\tan^2(s) + 1} ds.$$

Now change variables $\tan(s) = x$, $\frac{dx}{ds} = \sec^2(s)$:

$$\tan^{-1}(x) = \int_{x=\tan(0)}^{\tan^{-1}(x)} \frac{\sec^2(s)}{x^2 + 1} \frac{dx}{\sec^2(s)} = \int_0^x \frac{dx}{x^2 + 1} = \int_0^x \sum_{k=0}^{\infty} (-x^2)^k dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}.$$

One of the ways to write this as a matrix product is

$$\prod_{n \geq \frac{1}{2}} \begin{pmatrix} -\frac{n}{n+1} x^2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \tan^{-1}(x) \\ 0 & 1 \end{pmatrix}.$$

We can include this matrix in a path invariant system as a limiting case of our original I, J, K, N system. Again we drop the I and J dimensions, this time fixing i at $-t$ and j at $(z-1)t$:

$$K_{k,n} = \begin{pmatrix} \frac{(k-t)(k+t(z-1))}{k(k+n+t(z-2))} & \frac{1}{k} \\ 0 & 1 \end{pmatrix},$$

$$N_{k,n} = \begin{pmatrix} \frac{(n-t)(n+t(z-1))}{n(k+n+t(z-2))} & \frac{1}{n} \\ 0 & 1 \end{pmatrix}.$$

Next shift $k \rightarrow k + (1-z)t$, and $n \rightarrow n+t$, and "crosstie" by the matrix

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

which merely scales both UR elements by t :

$$K_{k,n} = \begin{pmatrix} \frac{k(k-tz)}{(k+n)(k+(1-z)t)} & \frac{t}{k+(1-z)t} \\ 0 & 1 \end{pmatrix},$$

$$N_{k,n} = \begin{pmatrix} \frac{n(n+t)z}{(k+n)(n+t)} & \frac{t}{n+t} \\ 0 & 1 \end{pmatrix}.$$

Finally, let $t \rightarrow \infty$:

$$K_{k,n} = \begin{pmatrix} \frac{kz}{(k+n)(z-1)} & \frac{1}{1-z} \\ 0 & 1 \end{pmatrix}, \quad N_{k,n} = \begin{pmatrix} \frac{nz}{k+n} & 1 \\ 0 & 1 \end{pmatrix}.$$

Then (with a nervous recheck for path-invariance, though we did nothing illegal) the N product, with $k_0 = 1$, $n_0 = 1/2$, $z = -x^2$

$$\prod_{n \geq 1/2} N_{1,n} = \prod_{n \geq \frac{1}{2}} \begin{pmatrix} -\frac{n}{n+1}x^2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\tan^{-1}(x)}{x} \\ 0 & 1 \end{pmatrix}$$

provides our arctan powerseries. The K product departing that same $(1, 1/2)$ origin is

$$\prod_{k \geq 1} K_{k,1/2} = \prod_{k \geq 1} \begin{pmatrix} \frac{2kx^2}{(2k+1)(x^2+1)} & \frac{1}{x^2+1} \\ 0 & 1 \end{pmatrix}.$$

When their UR sums converge (e.g., $|x| < 1$), these two products are equal by virtue of being the limits of two sides of a path-invariant rectangle whose other two sides grow negligible, as with the $\zeta(2)$ derivation. Testing through x^7 ,

$$\prod_{k=1}^4 K_{k,1/2} = \begin{pmatrix} \frac{128x^8}{315(x^2+1)^4} & \frac{16x^6}{35(x^2+1)^4} + \frac{8x^4}{15(x^2+1)^3} + \frac{2x^2}{3(x^2+1)^2} + \frac{1}{x^2+1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} O(x^8) & 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + O(x^8) \\ 0 & 1 \end{pmatrix}.$$

Now suppose $\tan^{-1}(x) = \sin^{-1}(y)$, i.e.,

$$\frac{x}{\sqrt{x^2+1}} = y, \quad x = \frac{y}{\sqrt{1-y^2}}.$$

Then the K product becomes

$$\prod_{k \geq 1} \begin{pmatrix} \frac{2k}{2k+1}y^2 & 1-y^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{1-y^2} \frac{\sin^{-1}(y)}{y} \\ 0 & 1 \end{pmatrix}$$

To get $\sin^{-1}(y)^2$, we want to integrate $\sin^{-1}(y)/\sqrt{1-y^2} dy$, so we must move all y -dependence to the UR of the "productand" by crosstyng by

$$\begin{pmatrix} \frac{1-y^2}{y^{2k-1}} & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} \frac{1-y^2}{y} & 0 \\ 0 & 1 \end{pmatrix} \prod_{k \geq 1} \begin{pmatrix} \frac{2k}{2k+1} & y^{2k-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{1-y^2} \frac{\sin^{-1}(y)}{y} \\ 0 & 1 \end{pmatrix}.$$

Transposing the orphaned crosstie,

$$\prod_{k \geq 1} \begin{pmatrix} \frac{2k}{2k+1} & y^{2k-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sin^{-1}(y)}{\sqrt{1-y^2}} \\ 0 & 1 \end{pmatrix}.$$

Integrating (from 0) both sides dy (remembering that upper-triangular matrix products are linear in their UR),

$$\prod_{k \geq 1} \begin{pmatrix} \frac{2k}{2k+1} & \frac{y^{2k}}{2k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sin^{-1}(y)^2}{2} \\ 0 & 1 \end{pmatrix},$$

gives us the powerseries for $\sin^{-1}(y)^2$. Now cross it by

$$\begin{pmatrix} \frac{y^{2k}}{3} & 0 \\ 0 & 1 \end{pmatrix},$$

to get rid of the y^{2k} (and sneak in a factor of 3):

$$\prod_{k \geq 1} \begin{pmatrix} \frac{2ky^2}{2k+1} & \frac{3}{2k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3\sin^{-1}(y)^2}{2y^2} \\ 0 & 1 \end{pmatrix}.$$

Finally, putting $y=1/2$, $\sin^{-1}(y) = \pi/6$, and

$$\prod_{k \geq 1} \begin{pmatrix} \frac{k}{2(2k+1)} & \frac{3}{2k} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi^2}{6} \\ 0 & 1 \end{pmatrix}.$$

But this is identically the matrix product we derived earlier by accelerating $\zeta(2)$. *Quod erat demonstrandum.*

Concluding sermonette

We only scratched the surface of what you can do with those I, J, K, N matrices. They can, e.g., derive Gauss's hypergeometric identity

$$\sum_{n=0}^{\infty} \frac{(a+n)!(b+n)!}{n!(1+c+n)!} = \frac{a!b!(c-a-b-1)!}{(c-a)!(c-b)!},$$

(for complex a, b, c), which generalizes a large number of binomial coefficient identities, and provides closed forms for many other sums. They can accelerate the convergence of many important series, providing numerical methods for special functions such as these two-bits-per-term products for the (complex) Beta function:

$$\prod_{k=0}^{\infty} \begin{pmatrix} \frac{(k+2-a)(k+b+1)(k+a+b)}{4(k+a)(k+\frac{b}{2}+2)(k+\frac{b+3}{2})} & \frac{3k^2+k(a+2b+3)+a+b}{b(b+1)(b+2)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & B(a, b) \\ 0 & 1 \end{pmatrix},$$

$$\prod_{k=0}^{\infty} \begin{pmatrix} \frac{(k+a)^2(k+b)^2}{4(k+1)^2(k+\frac{a+b}{2}+1)(k+\frac{1}{2}(a+b+1))} & \frac{3k^2+2k(a+b)+ab}{a+b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{B(a, b)} \\ 0 & 1 \end{pmatrix}.$$

When c is a positive integer, the N matrix product *telescopes*, giving the sum in closed form over any interval, not just $[0, \infty)$. E.g., when $c = 1$,

$$N_{a,b,1,n} = \begin{pmatrix} \frac{(a+n)(b+n)}{(n+1)(a+b+n+1)} & 1 \\ 0 & 1 \end{pmatrix} = : \begin{pmatrix} r_n & a_n \\ 0 & 1 \end{pmatrix}.$$

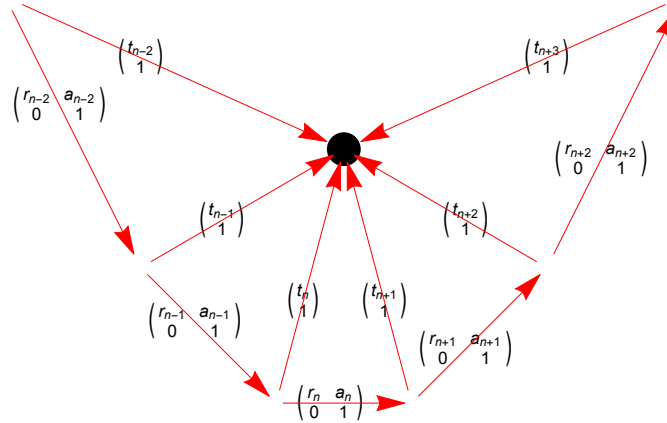
Telescoping in matrices means finding a function t_n such that

$$\begin{pmatrix} t_n \\ 1 \end{pmatrix} = \begin{pmatrix} r_n & a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_{n+1} \\ 1 \end{pmatrix}.$$

In this case, we have

$$\begin{pmatrix} t_n \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{n(a+b+n)}{a \ b} \\ 1 \end{pmatrix}.$$

Graphically, this says that multiplying any sequence of N matrices and then diving into the center via a t matrix is equivalent to simply diving straight in:



(The non-invertibility of the t matrices makes this a parabolic telescope focused on a true black hole, as opposed to a mere variable star of huge period.) If we choose the interval $j \leq n \leq k$,

$$\begin{pmatrix} -\frac{j(a+b+j)}{a \ b} \\ 1 \end{pmatrix} = \begin{pmatrix} \prod_{n=j}^k \frac{(a+n)(b+n)}{(n+1)(a+b+n+1)} & \sum_{n=j}^k \prod_{l=j}^{n-1} \frac{(a+l)(b+l)}{(l+1)(a+b+l+1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{(1+k)(1+a+b+k)}{a \ b} \\ 1 \end{pmatrix},$$

or

$$\begin{pmatrix} -\frac{j(a+b+j)}{a \ b} \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{n=j}^k \prod_{l=j}^{n-1} \frac{(a+l)(b+l)}{(l+1)(a+b+l+1)} - \frac{(k+1)(a+b+k+1)(a+j-j+k+1)(b+j-j+k+1)}{a \ b (j+1-j+k+1)(a+b+j+1-j+k+1)} \\ 1 \end{pmatrix}.$$

And if those I, J, K, N matrices fall short, no problem—they are merely a degenerate case of a six dimensional system of nontriangular 3×3 matrices, which can prove many generalized hypergeometric and continued fraction identities. *E.g.*, from the standard arctangent expansion

$$\frac{z}{\tan^{-1}(z)} = 1 + \frac{z^2}{3 + \frac{4z^2}{5 + \frac{9z^2}{\ddots}}},$$

we can get the arcsecant expansion

$$(s-1) \left(\frac{s+1}{\frac{\sqrt{s^2-1}}{\sec^{-1}(s)} - 1} + \frac{3}{2} \right) = 3 \cdot s + 0 + \frac{s^2-1}{4 \left(3 \cdot s + 2 + \frac{9(s^2-1)}{4 \left(3 \cdot s + 4 + \frac{25(s^2-1)}{4 \dots} \right)} \right)}.$$

And if these 3×3 matrices fall short, they are merely the $q \rightarrow 1$ limit of a more general system of 4×4 s, ... For a grand finale, we can exhibit a q -extension of Ramanujan's

$$\sum_{k \geq 0} \frac{42 \cdot k + 5}{2^{12 \cdot k + 4}} \binom{2 \cdot k}{k}^3 = \frac{1}{\pi},$$

so obtained, as

$$\begin{aligned} \sum_{k \geq 0} \frac{1 - q^{6 \cdot k + 1} - \frac{(1 - q^{6 \cdot k + 3}) q^{6 \cdot k + 1}}{(q^{2 \cdot k + 1} + 1)^3}}{1 - q^2} \frac{q^{6 \cdot k^2}}{\prod_{j=1}^k (1 + q^j)^6 \prod_{j=1}^{2 \cdot k} (1 + q^j)^3} \left(\frac{\prod_{j=1}^{2 \cdot k} (1 - q^j)}{\prod_{j=1}^k (1 - q^j)^2} \right)^3 \\ = \frac{1}{q + 1} \prod_{n \geq 1} \frac{(1 - q^{2 \cdot n - 1}) (1 - q^{2 \cdot n + 1})}{(1 - q^{2 \cdot n})^2} = \frac{2 \sqrt[4]{q}}{\pi (q + 1)} \sum_{n \geq 1} \frac{(1 - q)^{n-1}}{n} \prod_{n \geq 1} \coth^2 \left(\frac{\pi^2 n}{\log(q)} \right), \end{aligned}$$

where the middle product may be seen as (half the reciprocal of) a "q-Wallis product", equal to the last expression by Jacobi's imaginary ϑ transformation, whose $q \rightarrow 1$ limit is clearly $1/\pi$. (Note the numerical insignificance of

$$\prod_{n \geq 1} \coth^2 \left(\frac{\pi^2 n}{\log(.9)} \right) \simeq \prod_{n \geq 1} (1 - 10^{-80.7 \cdot n}),$$

even for $q = .9$.) Jacobi's transformation is a fairly fancy way to prove Wallis's product!