Commensurable triangles

The problem I would like to address is the following. It was posed and solved recently by Richard Parris [1], so what I have to say was inspired by him. But my solution is more direct.

We are given two positive integers h and k, which we may suppose are relatively prime, and we want to construct triangles in which the ratio of one angle to another is $h : k$.

This problem was motivated in part by the 4–5–6 triangle, in which one angle is twice another.

Let $\dfrac{p}{q}$ be any rational with p and q relatively prime, with

$$1 > \frac{p}{q} > \cos\frac{\pi}{h+k}.$$

Define α by

$$\cos\alpha = \frac{p}{q}.$$

Then $(h+k)\alpha < \pi$ and the triangle with sides

$$q^{h+k-1}\frac{\sin h\alpha}{\sin\alpha},\quad q^{h+k-1}\frac{\sin k\alpha}{\sin\alpha},\quad q^{h+k-1}\frac{\sin(h+k)\alpha}{\sin\alpha}$$

has integer sides, and angles $h\alpha$, $k\alpha$ and $\pi-(h+k)\alpha$. For example, if $h=2,\ k=1$, $\cos\dfrac{\pi}{3}=\dfrac{1}{2}$ and if we choose $p=3,\ q=4$, $\alpha=\cos^{-1}\dfrac{3}{4}$, then the triangle with sides

$$4^2\frac{\sin\alpha}{\sin\alpha}=16,\ \ 4^2\frac{\sin 2\alpha}{\sin\alpha}=32\cos\alpha=24,\ \ 4^2\frac{\sin 3\alpha}{\sin\alpha}=16(3-4\sin^2\alpha)=16(4\cos^2\alpha-1)=20,$$

or equivalently with sides 4, 6 and 5, has angles α, 2α and $\pi-3\alpha$.

First we note that a triangle with sides in the ratio $\sin A : \sin B : \sin(A+B)$ where $A>0$, $B>0$ and $A+B<\pi$ is similar to a triangle with angles $A,\ B$ and $\pi-(A+B)$, so itself has those angles. Now we need to prove that

$$q^{n-1}\frac{\sin n\alpha}{\sin\alpha}$$

is an integer.

We have $\cos\alpha = \dfrac{p}{q}$, so $\sin\alpha = \dfrac{r}{q}$, where $r = \sqrt{q^2 - p^2}$. So

$$e^{i\alpha} = \frac{p}{q} + i\frac{r}{q}, \quad e^{-i\alpha} = \frac{p}{q} - i\frac{r}{q},$$
$$e^{ni\alpha} = \frac{(p+ir)^n}{q^n}, \quad e^{-ni\alpha} = \frac{(p-ir)^n}{q^n},$$

$$\begin{aligned}
\frac{\sin n\alpha}{\sin\alpha} &= \frac{1}{q^{n-1}}\frac{(p+ir)^n - (p-ir)^n}{(p+ir)-(p-ir)} \\
&= \frac{1}{q^{n-1}}\frac{(p+ir)^n - (p-ir)^n}{2ri} \\
&= \frac{\Big(p^n + \binom{n}{1}p^{n-1}ir + \ \cdots\ \Big) - \Big(p^n - \binom{n}{1}p^{n-1}ir + \ \cdots\ \Big)}{q^{n-1}2ri} \\
&= \frac{\binom{n}{1}p^{n-1} - \binom{n}{3}p^{n-3}r^2 + \binom{n}{5}p^{n-5}r^4 - + \ \cdots}{q^{n-1}},
\end{aligned}$$

so

$$q^{n-1}\frac{\sin n\alpha}{\sin\alpha} = \binom{n}{1}p^{n-1} - \binom{n}{3}p^{n-3}(q^2-p^2) + \binom{n}{5}p^{n-5}(q^2-p^2)^2 - + \ \cdots$$

is clearly an integer.

I thought a numerical example might be in order.

Suppose $h = 3,\ k = 2$; $\cos\dfrac{\pi}{5} \approx 0.81$, so I choose $p = 9,\ q = 10$. Then the three sides of the triangle are (after dividing by 16) 1400, 1125 and 1111.

You, dear reader, may like to choose a different h, k, p and q and construct the corresponding triangle.

Reference

[1] Richard Parris, Commensurable triangles, The College Mathematics Journal 38 (2007), 345–355.