The ${}_3F_2[1]$ "Rosetta Stone,"

a path-invariant matrix system copied straight out of Macsyma's mac2mma function, which quaintly portrays matrices with List[List[...]]:

```
rosetta = replacemats[MapThread[List, {{i, j, k, n},
  {{{((-i) * (g + h + k) - k * (-g - h) + g * h + i^2) / ((i + n) * (-g - h + i + j - k - 1)),
    (g * h * k) / ((i + n) * (-g - h + i + j - k - 1)), (n * (j + n - 1)) /
     (-g - h + i + j - k - 1)}, {1 / (i + n), i / (i + n), 0}, {0, 0, 1}},
   {{((-j) * (g + h + k) - k * (-g - h) + g * h + j^2) / ((j + n) * (-g - h + i + j - k - 1)),
    (g * h * k) / ((j + n) * (-g - h + i + j - k - 1)), (n * (i + n - 1)) /
     (-g - h + i + j - k - 1)}, {1 / (j + n), j / (j + n), 0}, {0, 0, 1}},
   {{((k + n) * (g + h - i - j + k + 2)) / ((-i + k + 1) * (-j + k + 1)),
    (g * h * (k + n)) / ((-i + k + 1) * (-j + k + 1)), (n * (i + n - 1) * (j + n - 1)) /
     ((-i + k + 1) * (-j + k + 1))}, {-(((k + n) * (g + h - i - j + k + 2)) /
     (k * (-i + k + 1) * (-j + k + 1))), ((k + n) * (-(g * h) - k * (i + j - 2) - (1 - i) * j -
      i + k^2 + 1)) / (k * (-i + k + 1) * (-j + k + 1)),
    -((n * (i + n - 1) * (j + n - 1)) / (k * (-i + k + 1) * (-j + k + 1)))}, {0, 0, 1}},
   {{((g + n) * (h + n) * (k + n)) / ((n + 1) * (i + n) * (j + n)), 0, n},
    {0, ((g + n) * (h + n) * (k + n)) / ((n + 1) * (i + n) * (j + n)), 1}, {0, 0, 1}}}]]
```

These matrices facilitate the discovery and manipulation of hypergeometric identities.

$$\begin{pmatrix} i & \begin{pmatrix} \frac{i^2-(g+h+k)\,i+g\,h-(-g-h)\,k}{(-g-h+i+j-k-1)\,(i+n)} & \frac{g\,h\,k}{(-g-h+i+j-k-1)\,(i+n)} & \frac{n\,(j+n-1)}{-g-h+i+j-k-1} \\ \frac{1}{i+n} & \frac{i}{i+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} \frac{i^2-(g+h+k)\,j+g\,h-(-g-h)\,k}{(-g-h+i+j-k-1)\,(j+n)} & \frac{g\,h\,k}{(-g-h+i+j-k-1)\,(j+n)} & \frac{n\,(i+n-1)}{-g-h+i+j-k-1} \\ \frac{1}{j+n} & \frac{j}{j+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} \frac{(g+h-i-j+k+2)\,(k+n)}{(-i+k+1)\,(-j+k+1)} & \frac{g\,h\,(k+n)}{(-i+k+1)\,(-j+k+1)} & \frac{n\,(i+n-1)\,(j+n-1)}{(-i+k+1)\,(-j+k+1)} \\ -\frac{(g+h-i-j+k+2)\,(k+n)}{k\,(-i+k+1)\,(-j+k+1)} & \frac{\left(k^2-(i+j-2)\,k-g\,h-i-(1-i)\,j+1\right)(k+n)}{k\,(-i+k+1)\,(-j+k+1)} & -\frac{n\,(i+n-1)\,(j+n-1)}{k\,(-i+k+1)\,(-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & 0 & n \\ 0 & \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

Check path invariance:

```
picheck[%]
```

```
True
```

These matrices were found during months of noodling with Macsyma during the 1980s, but once their overall scheme was guessed, they, and their higher dimensional generalizations could be filled out in a matter of hours by the method of undetermined coefficients. Nowadays, with effective algorithms (Holonomics) for finding rational function solutions to functional equations, the entire system can be constructed more or less automatically, starting with just the n matrix.

$\prod_{n\geq 0}$(n matrix) computes $\{{}_3F_2[1], {}_3F_2[1], 1\}$ in the right column:

```
MProd[rosetta[[-1, 2]], {n, 0, ∞}]
```

Product::div : Product does not converge. ≫

These convergence warnings are an artifact of prematurely evaluated, partially instantiated expressions. They are probably not worth suppressing, since they're kind of a crude progress report.

Sum::div : Sum does not converge. ≫

$$\begin{pmatrix} \prod_{n=0}^{\infty} \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & 0 & \frac{g\,h\,k\,{}_3F_2(g+1,h+1,k+1;i+1,j+1;1)}{i\,j} \\ 0 & \prod_{n=0}^{\infty} \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & {}_3F_2(g,h,k;i,j;1) \\ 0 & 0 & 1 \end{pmatrix}$$

Element $_{2,2}$ of the n matrix,

```
mats[n][[2, 2]]
```

$$\frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)}$$

is the term ratio of the lower ${}_3F_2(1)$. The term ratio of the upper series is

```
mats[n][[1, 1]] * (n + 1) / n
```

$$\frac{(g+n)\,(h+n)\,(k+n)}{n\,(i+n)\,(j+n)}$$

due to the factor of n in the $_{1,3}$ element. For large n, this is, to first order,

```
Series[%, {n, ∞, 1}]
```

$$1+\frac{g+h-i-j+k}{n}+O\left(\left(\frac{1}{n}\right)^2\right)$$

for convergence of both sums, this must lag the term ratio of the harmonic series

```
# == Series[#, {n, ∞, 1}] &[n / (n + 1)]
```

$$\frac{n}{n+1}=1-\frac{1}{n}+O\left(\left(\frac{1}{n}\right)^2\right)$$

I.e.,

$$i+j>g+h+k+1$$

Actually, we only need

```
Assuming[Re[0 + i + j - (g + h + k)] > 1, MProd[mats[n], {n, 0, ∞}]]
```

Sum::div : Sum does not converge. ≫

$$\begin{pmatrix} 0 & 0 & \frac{g\,h\,k\,{}_3F_2(g+1,h+1,k+1;i+1,j+1;1)}{i\,j} \\ 0 & 0 & {}_3F_2(g,h,k;i,j;1) \\ 0 & 0 & 1 \end{pmatrix}$$

Important exemption! These convergence conditions are completely waived if g, h, or k is a non-positive integer, in which case the series simply terminates, no matter how fast the terms might have been growing.

Important exception! if both j and g, say, are non-posittive integers with j < g, the interpretation of the sum becomes ambiguous. Combinatorialists want it to terminate after -g terms, but this introduces discontinuity. E.g., as $\epsilon \to 0$ in

$${}_3F_2(h,k,g+\epsilon;i,j+\epsilon;z)$$

only the terms between -g and -j vanish. The series ``relights'' at term -j. It is tempting to reject this interpretation, since it smacks of 0/0, but it has a strong justification. Whenever a numerator parameter exceeds a denominator parameter by a positive integer k, e.g.,

$${}_3F_3(\ldots,b,a+k;\ldots,a,r;z)$$

the nth term contains a factor of

$$\frac{\Gamma(a+k+n)}{\Gamma(a+n)}=(a+k+n-1)*(a+k+n-1)*\ldots*(a+n)$$

a polynomial, whether or not a is a negative integer. And when it is, this makes it clear how only a burst of k terms can vanish.

These ${}_3F_2[1]$s and the n matrix behind them are symmetric in *g*,*h*,*k*. But *g* and *h* have no associated matrix, and are thus unavailable to explore. We can create the *h* matrix (and, perhaps later, *g*) by analogy with *k*:

```
RotateRight[updatemat[h, mats[k] /. {k → h, h → k}]]
```

$$\begin{pmatrix} h & \begin{pmatrix} \frac{(g+h-i-j+k+2)\,(h+n)}{(h-i+1)\,(h-j+1)} & \frac{g\,k\,(h+n)}{(h-i+1)\,(h-j+1)} & \frac{n\,(i+n-1)\,(j+n-1)}{(h-i+1)\,(h-j+1)} \\ -\frac{(g+h-i-j+k+2)\,(h+n)}{h\,(h-i+1)\,(h-j+1)} & \frac{\left(h^2-(i+j-2)\,h-i-(1-i)\,j-g\,k+1\right)(h+n)}{h\,(h-i+1)\,(h-j+1)} & -\frac{n\,(i+n-1)\,(j+n-1)}{h\,(h-i+1)\,(h-j+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ i & \begin{pmatrix} \frac{i^2-(g+h+k)\,i+g\,h-(-g-h)\,k}{(-g-h+i+j-k-1)\,(i+n)} & \frac{g\,h\,k}{(-g-h+i+j-k-1)\,(i+n)} & \frac{n\,(j+n-1)}{-g-h+i+j-k-1} \\ \frac{1}{i+n} & \frac{i}{i+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} \frac{i^2-(g+h+k)\,j+g\,h-(-g-h)\,k}{(-g-h+i+j-k-1)\,(j+n)} & \frac{g\,h\,k}{(-g-h+i+j-k-1)\,(j+n)} & \frac{n\,(i+n-1)}{-g-h+i+j-k-1} \\ \frac{1}{j+n} & \frac{j}{j+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} \frac{(g+h-i-j+k+2)\,(k+n)}{(-i+k+1)\,(-j+k+1)} & \frac{g\,h\,(k+n)}{(-i+k+1)\,(-j+k+1)} & \frac{n\,(i+n-1)\,(j+n-1)}{(-i+k+1)\,(-j+k+1)} \\ -\frac{(g+h-i-j+k+2)\,(k+n)}{k\,(-i+k+1)\,(-j+k+1)} & \frac{\left(k^2-(i+j-2)\,k-g\,h-i-(1-i)\,j+1\right)(k+n)}{k\,(-i+k+1)\,(-j+k+1)} & -\frac{n\,(i+n-1)\,(j+n-1)}{k\,(-i+k+1)\,(-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & 0 & n \\ 0 & \frac{(g+n)\,(h+n)\,(k+n)}{(n+1)\,(i+n)\,(j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

```
picheck[%]
```

```
True
```

(False would require some serious reality rot.)

Since this matrix system is path-invariant, we can freely create identities by multiplying matrices along the sides of closed loops. Unfortunately, besides n, none of the other matrices is triangular. Products along those axes just compute unfamiliar recurrences. An obvious remedy might be to concede one or more dimensions and fix their respective variables so as to annihilate element $_{2,1}$ in their respective matrices. But this produces only trivial identities. Until we recoordinatize the grid!

Dixon's Theorem (and friends)

```
recoord[k -> k - j]
```

$$\begin{pmatrix} i & \begin{pmatrix} \frac{(h-i)(i+j-k)+g(-h+i+j-k)}{(g+h-i-2j+k+1)(i+n)} & \frac{g\,h\,(j-k)}{(g+h-i-2j+k+1)(i+n)} & -\frac{n(j+n-1)}{g+h-i-2j+k+1} \\ \frac{1}{i+n} & \frac{j}{i+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} \frac{h(j-h)g^2+(j h^2+(2j^2+3ij-5kj+j+2k^2+i-2ik-k)h-(i+j-k)(4j^2+(2-4k)j+(k-1)k))g-(h-j)(i+j-k)(4j^2+(2-4k)j+(k-1)k)}{(g+h-i-2j+k)(g+h-i-2j+k+1)(j-k-n+1)(j+n)} & \frac{g\,h\,(j-k+1)(-k^2+ik+2jk+g(h-j)-hj-ij)}{(g+h-i-2j+k)(g+h-i-2j+k+1)(j-k-n+1)(j+n)} & -\frac{n(i+n-1)(-k^2+ik+2jk-nk+k-i-ij-2j+g(h-j-n+1)+in+2jn-h(j+n-1))}{(g+h-i-2j+k)(g+h-i-2j+k+1)(j-k-n+1)} \\ -\frac{k^2-2jk-kh+hj+jg(j-h)+i(j-k+1)}{(g+h-i-2j+k)(j-k-n+1)(j+n)} & -\frac{(g(h-j)+j(-h+i+2j-k))(j-k+1)}{(g+h-i-2j+k)(j-k-n+1)(j+n)} & \frac{n(i+n-1)}{(g+h-i-2j+k)(j-k-n+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} \frac{(g+h-i-2j+k+2)(-j+k+n)}{(-2j+k+1)(-i-j+k+1)} & \frac{g\,h\,(-j+k+n)}{(-2j+k+1)(-i-j+k+1)} & \frac{n(i+n-1)(j+n-1)}{(-2j+k+1)(-i-j+k+1)} \\ -\frac{(g+h-i-2j+k+2)(-j+k+n)}{(-2j+k+1)(k-j)(-i-j+k+1)} & -\frac{(g\,h-(i+j-k-1)(2j-k-1))(j-k-n)}{(j-k)(i+j-k-1)(2j-k-1)} & \frac{n(i+n-1)(j+n-1)}{(j-k)(i+j-k-1)(2j-k-1)} \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(g+n)(h+n)(-j+k+n)}{(n+1)(i+n)(j+n)} & 0 & n \\ 0 & \frac{(g+n)(h+n)(-j+k+n)}{(n+1)(i+n)(j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ h & \begin{pmatrix} \frac{(g+h-i-2j+k+2)(h+n)}{(h-i+1)(h-j+1)} & \frac{g(k-j)(h+n)}{(h-i+1)(h-j+1)} & \frac{n(i+n-1)(j+n-1)}{(h-i+1)(h-j+1)} \\ -\frac{(g+h-i-2j+k+2)(h+n)}{h(h-i+1)(h-j+1)} & \frac{(h^2-(i+j-2)h+i(j-1)+gj-jg-gk+1)(h+n)}{h(h-i+1)(h-j+1)} & -\frac{n(i+n-1)(j+n-1)}{h(h-i+1)(h-j+1)} \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

This changed all occurrences of *k* to be *k-j*, and replaced the *j* matrix by *j* times the inverse of the *k* matrix, to make it decrement *k* while bumping *j*. (This could only de-triangularize the *n* matrix if *n* appeared on the right of a "→".) Now element $_{2,1}$ of the *j* matrix has a numerator linear in *j* with coefficients

```
CoefficientList[Numerator[mats[j][[2, 1]]], j]
```

$\{g\,h+i\,k-i-k^2+k,\ -g-h-i+2\,k-1\}$

which we can annihilate by sacrificing *k* and *i*:

```
Solve[0 == # & /@ {g h - i + k + i k - k^2, -1 - g - h - i + 2 k}, {k, i}]
```

$\{\{k\to g+1,\ i\to g-h+1\},\ \{k\to h+1,\ i\to -g+h+1\}\}$

Either solution will do.

```
recoord @@ %[[1]]
```

$$\begin{pmatrix} i & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} -\frac{(h-j)(g^2-3jg+g+j(2j-1))}{(g+2h-2j+1)(g-j+n)(j+n)} & -\frac{g\,h\,(g-j)(h-j)}{(g+2h-2j)(g+2h-2j+1)(g-j+n)(j+n)} & -\frac{n(g-h+n)(g(n-1)+(h-j)(2n-1))}{(g+2h-2j)(g+2h-2j+1)(g-j+n)} \\ 0 & \frac{(g-2j)(g-j)(j-h)}{(g+2h-2j)(g-j+n)(j+n)} & -\frac{n(g-h+n)}{(g+2h-2j)(g-j+n)} \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(g+n)(h+n)(g-j+n+1)}{(n+1)(g-h+n+1)(j+n)} & 0 & n \\ 0 & \frac{(g+n)(h+n)(g-j+n+1)}{(n+1)(g-h+n+1)(j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ h & \begin{pmatrix} \frac{(g+2h-2j+3)(g-h+n)(h+n)}{(g-2h-1)h(h-j+1)} & \frac{g(g-j+1)(g-h+n)(h+n)}{(g-2h-1)(g-2h)h(h-j+1)} & \frac{n(g-h+n)(j+n-1)(-2nh+h+gn)}{(g-2h-1)(g-2h)h(h-j+1)} \\ 0 & \frac{(g+2h-2j+2)(g-h+n)(h+n)}{(g-2h)h(h-j+1)} & \frac{n(g-h+n)(j+n-1)}{(g-2h)h(h-j+1)} \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

``Miraculously'', the h matrix has also triangularized. Hypergeometric wonks will recognize the middle row of the n matrix as the term ratio of Dixon's theorem. Taking the infinite matrix product computes a "contiguous Dixon's" (Mason's?) and actual Dixon's $_3F_2[1]$s in the rightmost column:

```
Assuming[Re[1 + 1 + g - h + j - (g + h + 1 + g - j) - 1] > 1, ToPFQ[MProdi[mats[n], {n, 0, ∞}]]]
```

$$\begin{pmatrix} 0 & 0 & \frac{g\,h\,(g-j+1)\,\text{HypergeometricPFQ}(\{g+1,h+1,g-j+2\},\{g-h+2,j+1\},1)}{(g-h+1)\,j} \\ 0 & 0 & \text{HypergeometricPFQ}(\{g,h,g-j+1\},\{g-h+1,j\},1) \\ 0 & 0 & 1 \end{pmatrix}$$

(Note how Assuming the convergence condition vanished the infinite products.) MProd is a function written mostly by Julian Ziegler Hunts which converts products of triangular matrices to matrices of sums and ordinary products. MProdi Inactivates these sums, to prevent idiocy on the part of *Mathematica*'s "simplifier". And I had to write my own ToPFQ to avert a single sum turning into dozens. Or hundreds! Unfortunately, the Inactivation ruins the pretty display. Let's risk Activating:

```
Activate[%]
```

$$\begin{pmatrix} 0 & 0 & \frac{g\,h\,(g-j+1)\,{}_3F_2(g+1,h+1,g-j+2;g-h+2,j+1;1)}{(g-h+1)\,j} \\ 0 & 0 & \frac{(g+1)_{-h}\left(j-\frac{g}{2}\right)_{-h}}{\left(\frac{g}{2}+1\right)_{-h}(j)_{-h}} \\ 0 & 0 & 1 \end{pmatrix}$$

Well, that's not so bad. It turned the Dixon sum into four Pochhammer symbols. The only problems: I didn't ask it to, and it (inexcusably) doesn't know the contiguous sums, else it would have overreacted to the "Mason" sum as well. Let's see if *Mathematica* got Dixon's Theorem right, and also derive "Mason's Theorem", by closing the infinite rectangular contour (n=0,j=j) ↗(∞,∞). We need to tweak these matrices so as to keep them finite when certain parameters blow up. Our plan is to equate the matrix product along the path (n=0,j=j) to (n=∞,j=j) to (n=∞,j=∞) (=BOTTOM.RIGHT) with the path (n=0,j=j) to (n=0,j=∞) to (n=∞,j=∞) (=LEFT.TOP). The rigorous, somewhat unpleasant way to do this is to close at ($n = n_{\max}$, $j=j_{\max}$) instead of (n=∞, j=∞) and then carefully let ($n_{\max}\to\infty$, $j_{\max}\to\infty$), producing true but perhaps varied identities depending on how the limits are taken.

But if we can tweak the *j* matrix to remain finite as $n\to\infty$, we can merrily slap together an infinite rectangle, and extract identities by equating corresponding matrix elements across the equation. Nothing impedes this except, while computing the RIGHT edge, the right column of *j* is $O(n^2)$ when we want $O(1)$. Perhaps the cleanest way to fix this is to sidestep by the matrix

$$\begin{pmatrix} (g+n)(g-h+n) & 0 & 0 \\ 0 & g-h+n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This will have the mildly annoying effect of scaling our $_3F_2[1]$ s by (g+n) (g-h+n) and g-h+n respectively. A slick trick to remedy this is to normalize by the n=0 case:

```
%147.Inverse[%147 /. n → 0] // Factor
```

$$\begin{pmatrix} \frac{(g+n)(g-h+n)}{g(g-h)} & 0 & 0 \\ 0 & \frac{g-h+n}{g-h} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But we can be even cleverer by noticing that the "Mason's Theorem" $_3F_2[1]$ was born with a coefficient of

$$\frac{g\,h\,(g-j+1)}{j\,(g-h+1)}$$

as an artifact of $_pF_q$ notation and path-invariance. With a fancier sidestep matrix,

```
MapAt[# * g * h * (g - j + 1) / (g - h + 1) / j &, %, {1, 1}]
```

$$\begin{pmatrix} \frac{h(g-j+1)(g+n)(g-h+n)}{(g-h)(g-h+1)j} & 0 & 0 \\ 0 & \frac{g-h+n}{g-h} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can scale that out, too:

```
sidestep[%157]
```

$$\begin{pmatrix} h & \begin{pmatrix} \frac{(g-h+1)(h+1)(g+2h-2j+3)(g-h+n-1)(h+n)}{(g-2h-1)(g-h-1)h^2(h-j+1)} & \frac{g(g-h)(g-h+1)j(g-h+n-1)(h+n)}{(g-2h-1)(g-2h)(g-h-1)h^2(h-j+1)(g+n)} & \frac{(g-h)(g-h+1)jn(j+n-1)(-2nh+h+gn)}{(g-2h-1)(g-2h)h^2(g-j+1)(h-j+1)(g+n)} \\ 0 & \frac{(g-h)(g+2h-2j+2)(g-h+n-1)(h+n)}{(g-2h)(g-h-1)h(h-j+1)} & \frac{(g-h)n(j+n-1)}{(g-2h)h(h-j+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ i & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} \frac{(g-2j+1)(g-j)^2 j(j-h)}{(g+2h-2j+1)(g-j+1)(j+1)(g-j+n)(j+n)} & -\frac{g(g-h+1)(g-j)(h-j)j}{(g+2h-2j)(g+2h-2j+1)(g-j+1)(g+n)(g-j+n)(j+n)} & -\frac{(g-h)(g-h+1)jn(g(n-1)+(h-j)(2n-1))}{h(g+2h-2j)(g+2h-2j+1)(g-j+1)(g+n)(g-j+n)} \\ 0 & \frac{(g-2j)(g-j)(j-h)}{(g+2h-2j)(g-j+n)(j+n)} & \frac{(h-g)n}{(g+2h-2j)(g-j+n)} \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(g+n+1)(h+n)(g-j+n+1)}{(n+1)(g-h+n)(j+n)} & 0 & \frac{(g-h)(g-h+1)jn}{h(g-j+1)(g+n)(g-h+n)} \\ 0 & \frac{(g+n)(h+n)(g-j+n+1)}{(n+1)(g-h+n)(j+n)} & \frac{g-h}{g-h+n} \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

Now the j matrix remains finite as $n\to\infty$:

```
Factor[Limit[mats[j], n → ∞]]
```

$$\begin{pmatrix} 0 & 0 & -\frac{(g-h)(g-h+1)j}{h(g+2h-2j+1)(g-j+1)} \\ 0 & 0 & -\frac{g-h}{g+2h-2j} \\ 0 & 0 & 1 \end{pmatrix}$$

and the infinite product matrix for the leg (n=∞,j=j) to (n=∞,j=∞) is conveniently degenerate:

```
RIGHT = MProd[%, {j, j, ∞}]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This gets multiplied on the left by the BOTTOM (n=0,j=j) to (n=∞,j=j) matrix

```
BOTTOM = Assuming[1 + g - h + j > g + 1 + h + g - j + 1, ToPFQ[MProdi[mats[n], {n, 0, ∞}]]]
```

$$\begin{pmatrix} 0 & 0 & \text{HypergeometricPFQ}(\{g+1,h+1,g-j+2\},\{g-h+2,j+1\},1) \\ 0 & 0 & \text{HypergeometricPFQ}(\{g,h,g-j+1\},\{g-h+1,j\},1) \\ 0 & 0 & 1 \end{pmatrix}$$

Here's a way to thwart *Mathematica*'s imposition of Dixon's Theorem:

```
BOTTOM = Activate[MapAt[HoldForm, %, {2, 3, 3}]]
```

$$\begin{pmatrix} 0 & 0 & {}_3F_2(g+1,h+1,g-j+2;g-h+2,j+1;1) \\ 0 & 0 & {}_3F_2(g,h,g-j+1;g-h+1,j;1) \\ 0 & 0 & 1 \end{pmatrix}$$

I.e., wrap a "freezebomb" around the final "1" in the $_3F_2(g,\ h,\ g-j+1;\ g-h+1,\ j;\ 1)$.

The factors of n in the $_{1,3}$ and $_{2,3}$ elements of the j matrix leave our LEFT (n=0,j=j) to (n=0,j=∞) leg almost diagonal:

```
mats[j] /. n → 0
```

$$\begin{pmatrix} \frac{(g-2j+1)(g-j)(j-h)}{(g+2h-2j+1)(g-j+1)(j+1)} & -\frac{(g-h+1)(h-j)}{(g+2h-2j)(g+2h-2j+1)(g-j+1)} & 0 \\ 0 & \frac{(g-2j)(j-h)}{(g+2h-2j)j} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When does its product converge?

```
Series[%155, {j, ∞, 2}]
```

$$\begin{pmatrix} 1+\frac{hg+2g+h+2}{2j^2}+O\left(\left(\frac{1}{j}\right)^3\right) & \frac{-g+h-1}{4j^2}+O\left(\left(\frac{1}{j}\right)^3\right) & 0 \\ 0 & 1+\frac{gh}{2j^2}+O\left(\left(\frac{1}{j}\right)^3\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Always.
But critcal to the existence of a closed form (i.e., infinite product) for Dixon's Theorem (and most others) is the presence of those factors of n in the (LEFT) j matrix, which are set to zero to connect with the (BOTTOM) n product starting at n=0. Corresponding to those numerator factors of n are the factors of n+1 in the denominators of the diagonal elements (essentially the term ratios) of the n matrix. These factors cancel in the path-invariance subexpression (n matrix(j)).(j matrix(n+1)). This denominator factor of n+1 corresponds to the "mandatory hidden 1" in ${}_pF_q[z]$, whose term ratio is actually a quotient of polynomials of degrees p and q+1. In the uncommon case without this denominator n+1, an artificial 1 must be adjoined to the numerator parameters to cancel it. But why is this the uncommon case? What is so magic about that factor of n+1 that the designers of ${}_pF_q$ notation incuded it automatically? Simple: a factor of 1/(n+1) in the nth term ratio means a factor of 1/n! in the nth term. The sums are from n = 0. What are the preceding terms? 1/(-1)!, 1/(-2)!, ..., I.e., all 0. The sum "terminates on the left". 0 is its natural starting point. A series is unlikely to add up to anything useful or recognizable if some number of initial terms are excluded.

Representation of series as explicit matrix products avoids these awkward hidden 1 problems, as well as the need to list the roots of the two polynomials, which can get messy.

```
LEFT = MProd[mats[j] /. n → 0, {j, j, ∞}] // tim
```

Sum::div : Sum does not converge. ≫

```
62.172284,3
```

Idiot *Mathematica* returned a sum of 341 HypergeometricPFQs! I had to write MProdi to Inactivate the sums, plus my own Sum to PFQ converter:

```
LEFT = Activate[ToPFQ[MProdi[mats[j] /. n → 0, {j, J, ∞}]] /. J → j]
```

$$\begin{pmatrix} -\frac{\Gamma(j+1)\Gamma\left(-\frac{g}{2}-h+j-\frac{1}{2}\right)}{(g-j+1)\Gamma\left(-\frac{g}{2}+j-\frac{1}{2}\right)\Gamma(j-h)} & -\frac{(g-h+1)(h-j)\,j\,\Gamma(j)\,\Gamma\left(-\frac{g}{2}-h+j+1\right){}_3F_2\left(1,-\frac{g}{2}+j-\frac{1}{2},-\frac{g}{2}-h+j,-\frac{g}{2}-h+j+1,-\frac{g}{2}-h+j+\frac{1}{2};1\right)}{(g+2h-2j)(g+2h-2j+1)(g-j+1)\Gamma\left(-\frac{g}{2}+j+1\right)\Gamma(-h+j+1)} & 0 \\ 0 & \frac{\Gamma(j)\,\Gamma\left(-\frac{g}{2}-h+j\right)}{\Gamma\left(j-\frac{g}{2}\right)\Gamma(j-h)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(The J→j was because MProdi got confused when by j being both the summation index and starting value.)

Finally, the TOP (n=0,j=∞) leg has matrix

```
Factor[Limit[mats[n], j → ∞]]
```

$$\begin{pmatrix} -\frac{(g+n+1)(h+n)}{(n+1)(g-h+n)} & 0 & -\frac{(g-h)(g-h+1)\,n}{h(g+n)(g-h+n)} \\ 0 & -\frac{(g+n)(h+n)}{(n+1)(g-h+n)} & \frac{g-h}{g-h+n} \\ 0 & 0 & 1 \end{pmatrix}$$

This produces two ${}_2F_1[-1]$s:

```
TOP = Assuming[1 + g - h > g + 1 + h, ToPFQ[MProdi[%, {n, 0, ∞}]]]
```

$$\begin{pmatrix} 0 & 0 & \text{HypergeometricPFQ}(\{g+1,h+1\},\{g-h+2\},-1) \\ 0 & 0 & \text{HypergeometricPFQ}(\{g,h\},\{g-h+1\},-1) \\ 0 & 0 & 1 \end{pmatrix}$$

Inexcusably, *Mathematica* can only do one of them:

```
Activate[%]
```

$$\begin{pmatrix} 0 & 0 & {}_2F_1(g+1,h+1;g-h+2;-1) \\ 0 & 0 & \frac{2^{-g}\sqrt{\pi}\,\Gamma(g-h+1)}{\Gamma\left(\frac{g}{2}+\frac{1}{2}\right)\Gamma\left(\frac{g}{2}-h+1\right)} \\ 0 & 0 & 1 \end{pmatrix}$$

The loop closure is, at last,

```
BOTTOM.RIGHT == Activate[LEFT.TOP]
```

$$\begin{pmatrix} 0 & 0 & {}_3F_2(g+1,h+1,g-j+2;g-h+2,j+1;1) \\ 0 & 0 & {}_3F_2(g,h,g-j+1;g-h+1,j;1) \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & -\frac{\Gamma(j+1)\Gamma\left(-\frac{g}{2}-h+j-\frac{1}{2}\right){}_2F_1(g+1,h+1;g-h+2;-1)}{(g-j+1)\Gamma\left(-\frac{g}{2}+j-\frac{1}{2}\right)\Gamma(j-h)} - \frac{2^{-g}(g-h+1)(h-j)\,j\sqrt{\pi}\,\Gamma(g-h+1)\Gamma(j)\Gamma\left(-\frac{g}{2}-h+j+1\right){}_3F_2\left(1,-\frac{g}{2}+j-\frac{1}{2},-\frac{g}{2}-h+j,-\frac{g}{2}+j+1,-\frac{g}{2}-h+j+\frac{1}{2};1\right)}{(g+2h-2j)(g+2h-2j+1)(g-j+1)\Gamma\left(\frac{g}{2}+\frac{1}{2}\right)\Gamma\left(\frac{g}{2}-h+1\right)\Gamma\left(-\frac{g}{2}+j+1\right)\Gamma(-h+j+1)} \\ 0 & 0 & \frac{2^{-g}\sqrt{\pi}\,\Gamma(g-h+1)\Gamma(j)\Gamma\left(-\frac{g}{2}-h+j\right)}{\Gamma\left(\frac{g}{2}+\frac{1}{2}\right)\Gamma\left(\frac{g}{2}-h+1\right)\Gamma\left(j-\frac{g}{2}\right)\Gamma(j-h)} \\ 0 & 0 & 1 \end{pmatrix}$$

with a perhaps disappointing deficiency in elegance. Despair not. First, check the second rows for equality of *Mathematica*'s Pochhammer expression with our Γ function expression for the ordinary Dixon's Theorem:

```
FullSimplify[ReleaseHold[#[[2]] & /@ %]]
```

```
True
```

Dixon's Theorem traditionally written:

```
#[[2, 3]] & /@ %% /. {g → a, h → b, j → a - c + 1}
```

$${}_3F_2(a,b,c;a-b+1,a-c+1;1) = \frac{\sqrt{\pi}\,2^{-a}\,\Gamma(a-b+1)\,\Gamma(a-c+1)\,\Gamma\left(\frac{a}{2}-b-c+1\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)\Gamma\left(\frac{a}{2}-c+1\right)\Gamma(a-b-c+1)}$$

But what of "Mason", the identity of the top rows? Note that because the ${}_2F_1[-1]$ is independent of j, we can choose j to produce numerous identities for it. E.g., for j=g+2,

```
#[[1, 3]] & /@ Activate[ReleaseHold[BOTTOM.RIGHT == LEFT.TOP]] /. {g → g - 1, h → h - 1, j → g + 2} //
  FunctionExpand // FullSimplify // dereg
```

$$\frac{1}{2+g-h}\left(\frac{(1+g)(2+g-2h)(4+g-2h)\sqrt{\pi}}{-1+h} + 8\,\text{Gamma}\left[\frac{3+g}{2}\right]\text{Gamma}\left[3+\frac{g}{2}-h\right]\left(-\frac{\sqrt{\pi}}{(-1+h)\,\text{Gamma}\left[\frac{g}{2}\right]\text{Gamma}\left[\frac{1}{2}(3+g-2h)\right]} + \frac{2^g\,\text{Hypergeometric2F1}[g,h,2+g-h,-1]}{\text{Gamma}[2+g-h]}\right)\right) == 0$$

(where we also canonicalized g→g - 1, h→h - 1. Repeated invocations of "dereg" are to undo FullSimplify's insistence on awkwardly "regularizing" our ${}_pF_q$ functions.) Disclosure: the FunctionExpand saved us the trouble of summing a "telescoping" ${}_3F_2(1)$, omitted for brevity, since such sums can be done completely automatically. The sum could have been preempted entirely by annihilating the ${}_{1,2}$ element of the j matrix with a sidestep computed by what is now a routine process. This changes the top matrix row to compute a linear combination of Dixon's sum and one contiguous, completely decoupled from the second, Dixon row. (Optionally breaking the system into two independent systems of 2x2s.)

Resuming, we can now solve for the contiguous ${}_2F_1(-1)$:

```
Solve[%, Hypergeometric2F1[g, h, 2 + g - h, -1]]
```

$$\left(\text{Null}\left(\left({}_2F_1(g,h;g-h+2;-1) \to \frac{2^{-g-3}(g-h+2)\left(\frac{8\sqrt{\pi}\,\Gamma\left(\frac{g+3}{2}\right)\Gamma\left(\frac{g}{2}-h+3\right)}{(g-h+2)(h-1)\Gamma\left(\frac{g}{2}\right)\Gamma\left(\frac{1}{2}(g-2h+3)\right)} - \frac{(g+1)(g-2h+2)(g-2h+4)\sqrt{\pi}}{(g-h+2)(h-1)}\right)\Gamma(g-h+2)}{\Gamma\left(\frac{g+3}{2}\right)\Gamma\left(\frac{g}{2}-h+3\right)}\right)\right)\right)$$

It is presumably a ridiculous bug that *Mathematica* has multiplied this solution by "Null".

```
FullSimplify[%[[1, 1]] / Null]
```

$${}_2F_1(g,h;g-h+2;-1) \to \frac{\sqrt{\pi}\,2^{-g}\left(\frac{1}{\Gamma\left(\frac{g}{2}\right)\Gamma\left(\frac{1}{2}(g-2h+3)\right)} - \frac{1}{\Gamma\left(\frac{g+1}{2}\right)\Gamma\left(\frac{g}{2}-h+1\right)}\right)\Gamma(g-h+2)}{h-1}$$

Which *Mathematica* should have known all along. Now we can simplify "Mason":

```
#[[1, 3]] & /@ %266 /. {g → g - 1, h → h - 1} /. %285 // FullSimplify // dereg
```

$${}_3F_2(g,h,g-j+1;g-h+2,j+1;1) = \frac{\sqrt{\pi}\,2^{-g}\,\Gamma(j+1)\,\Gamma(g-h+2)\left(\frac{\Gamma\left(-\frac{g}{2}-h+j+1\right)}{\Gamma\left(\frac{g+1}{2}\right)\Gamma\left(\frac{g}{2}-h+1\right)\Gamma\left(j-\frac{g}{2}\right)} - \frac{\Gamma\left(-\frac{g}{2}-h+j+\frac{3}{2}\right)}{\Gamma\left(\frac{g}{2}\right)\Gamma\left(\frac{1}{2}(g-2h+3)\right)\Gamma\left(-\frac{g}{2}+j+\frac{1}{2}\right)}\right)}{(h-1)(g-j)\,\Gamma(-h+j+1)}$$

Traditionalizing the parameters:

```
% /. j -> g - c + 1 /. g → a /. h → b
```

$${}_3F_2(a,b,c;a-b+2,a-c+2;1) = \frac{\sqrt{\pi}\,2^{-a}\,\Gamma(a-b+2)\,\Gamma(a-c+2)\left(\frac{\Gamma\left(\frac{a}{2}-b-c+2\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)\Gamma\left(\frac{a}{2}-c+1\right)} - \frac{\Gamma\left(\frac{a}{2}-b-c+\frac{5}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1}{2}(a-2b+3)\right)\Gamma\left(\frac{a}{2}-c+\frac{3}{2}\right)}\right)}{(b-1)(c-1)\,\Gamma(a-b-c+2)}$$

Now that we have these **pairs of contiguous valuations** for both Mason & Dixon, and the two limiting ${}_2F_1[-1]$s, we are in fat city, because the Rosetta matrix system lets us sidestep to a closed form for *any* ${}_pF_q$ with parameters congruent to these mod 1.

Besides n+1, there is another j-free factor, n-g-h in the denominator of the n matrix term ratio. This seems to suggest that we could get another closed form with the recoord transformation n -> n+g-h+1, which will put a new n+1 in the term ratio denominators, but strangely, fails to create a new n in the right column j numerators. This is merely because the n matrix needs first to be put in more nearly ratio form with a sidestep by

$$\begin{pmatrix} \frac{1}{g-h+n} & 0 & 0 \\ 0 & \frac{1}{g-h+n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This will change the denominator n+g-h to n+g-h+1, and multiply the top two elements of the j matrix right column by n + g - h. Then we can recoord[n -> n+g-h+1] and proceed as with the Dixon derivation. Unfortunately, this just rederives another form of Dixon's theorem due to symmetry among the parameters. But in less symmetrical situations, such maneuvers can actually produce additional identities.

Returning to the original, six parameter Rosetta system, we could build a general four parameter system describing ${}_2F_1[z]$ by letting i and g, say, →∞ in ratio z. But before that, a pleasant surprise.

Dougall's Theorem

A generalization of Dixon's ${}_3F_2[1]$ is http://mathworld.wolfram.com/DougallsTheorem.html ("very well poised") ${}_5F_4[1]$. We can get it from Rosetta, which seems to describe only ${}_3F_2[1]$ s! Just to be greedy, we enable *h* via symmetry with *k*, as before. But instead of `recoord[k -> k - j]`, try

```
replacemats[rosetta]; updatemat[h, mats[k] /. {k → h, h → k}]; recoord[j → j - i]
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{(g(h(i-j+k+1)-(i-j+1)(i-k))-(h-i)(i-j+1)(i-k))(i-j-n+1)}{(g+i-j+1)(h+i-j+1)(i-j+k+1)(i+n)} & -\frac{g\,h(2\,i-j+1)\,k\,(i-j-n+1)}{(g+i-j+1)(h+i-j+1)(i-j+k+1)(i+n)} & \frac{(2\,i^2-3\,(j-1)\,i+(j-1)^2)(i-j-n+1)\,n}{(g+i-j+1)(h+i-j+1)(i-j+k+1)} \\ \frac{(2\,i-j+1)(g+h-j+k+1)(i-j-n+1)}{(g+i-j+1)(h+i-j+1)(i-j+k+1)(i+n)} & \frac{(i^3+(g+h-2\,j+k+2)\,i^2+(g(h-j+k+1)-(h-j+1)(j-k-1))\,i-g\,h\,k)(i-j-n+1)}{(g+i-j+1)(h+i-j+1)(i-j+k+1)(i+n)} & \frac{(2\,i-j+1)(i-j-n+1)\,n}{(g+i-j+1)(h+i-j+1)(i-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
j \quad \begin{pmatrix} -\frac{(h+i-j)(i-j+k)+g(h+i-j+k)}{(g+h-j+k+1)(-i+j+n)} & -\frac{g\,h\,k}{(g+h-j+k+1)(-i+j+n)} & -\frac{n(i+n-1)}{g+h-j+k+1} \\ \frac{1}{-i+j+n} & \frac{j-i}{-i+j+n} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
k \quad \begin{pmatrix} \frac{(g+h-j+k+2)(k+n)}{(-i+k+1)(i-j+k+1)} & \frac{g\,h(k+n)}{(-i+k+1)(i-j+k+1)} & \frac{n(i+n-1)(-i+j+n-1)}{(-i+k+1)(i-j+k+1)} \\ -\frac{(g+h-j+k+2)(k+n)}{k(-i+k+1)(i-j+k+1)} & -\frac{(g\,h+(i-k-1)(i-j+k+1))(k+n)}{k(-i+k+1)(i-j+k+1)} & \frac{(i-j-n+1)\,n(i+n-1)}{k(-i+k+1)(i-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} \frac{(g+n)(h+n)(k+n)}{(n+1)(i+n)(-i+j+n)} & 0 & n \\ 0 & \frac{(g+n)(h+n)(k+n)}{(n+1)(i+n)(-i+j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
h \quad \begin{pmatrix} \frac{(g+h-j+k+2)(h+n)}{(h-i+1)(h+i-j+1)} & \frac{g\,k(h+n)}{(h-i+1)(h+i-j+1)} & \frac{n(i+n-1)(-i+j+n-1)}{(h-i+1)(h+i-j+1)} \\ -\frac{(g+h-j+k+2)(h+n)}{h(h-i+1)(h+i-j+1)} & \frac{(h^2-(j-2)\,h-i^2+i\,j-j-g\,k+1)(h+n)}{h(h-i+1)(h+i-j+1)} & \frac{(i-j-n+1)\,n(i+n-1)}{h(h-i+1)(h+i-j+1)} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

The i matrix is triangular merely when g+h-j+k+1=0. Let's sacrifice h for that:

```
recoord[h → j - k - g - 1]
```

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity encountered. ≫

Infinity::indet : Indeterminate expression ComplexInfinity + ComplexInfinity + ComplexInfinity encountered. ≫

Uh-oh, insufficient intermediate simplification. Try again with "one" instead of 1.

```
replacemats[rosetta]; updatemat[h, mats[k] /. {k → h, h → k}];
recoord[j → j - i, h → j - k - g - one]
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{(i-j-n+1)\{(i-j+k+1)\,g^2-(i-j+k+1)(j-k-\text{one})\,g-(i-j+1)(i-k)(i-j+k+\text{one})\}}{(g+i-j+1)(i-j+k+1)(i+n)(g-i+k+\text{one}-1)} & -\frac{g(2\,i-j+1)\,k\,(i-j-n+1)(g-j+k+\text{one})}{(g+i-j+1)(i-j+k+1)(i+n)(g-i+k+\text{one}-1)} & -\frac{(2\,i^2-3\,(j-1)\,i+(j-1)^2)(i-j-n+1)\,n}{(g+i-j+1)(i-j+k+1)(g-i+k+\text{one}-1)} \\ \frac{(2\,i-j+1)(i-j-n+1)(\text{one}-1)}{(g+i-j+1)(i-j+k+1)(i+n)(g-i+k+\text{one}-1)} & \frac{(i-j-n+1)\{-i^3+(j+\text{one}-2)\,i^2+(g^2+(-j+k+\text{one})\,g-(j-k-1)(k+\text{one}-1))\,i-g\,k(g-j+k+\text{one})\}}{(g+i-j+1)(i-j+k+1)(i+n)(g-i+k+\text{one}-1)} & -\frac{(2\,i-j+1)(i-j-n+1)\,n}{(g+i-j+1)(i-j+k+1)(g-i+k+\text{one}-1)} \\ 0 & 0 & 1 \end{pmatrix} \\
j \quad \begin{pmatrix} \frac{(g+i-j+k)(g-j+k-n+\text{one})}{(i-j-n)(g+i-j+k+\text{one}-1)} & \frac{g\,k(g-j+k-n+\text{one})}{(i-j-n)(g+i-j+k+\text{one}-1)} & \frac{n(i+n-1)}{g+i-j+k+\text{one}-1} \\ -\frac{(\text{one}-1)(g-j+k-n+\text{one})}{(i-j-n)(g-j+k+\text{one})(g+i-j+k+\text{one}-1)} & -\frac{(-g+j-k+n-\text{one})(i^2+(g-2\,j+k+\text{one}-1)\,i+j^2+g\,k-j(g+k+\text{one}-1))}{(i-j-n)(g-j+k+\text{one})(g+i-j+k+\text{one}-1)} & \frac{n(i+n-1)}{(g-j+k+\text{one})(g+i-j+k+\text{one}-1)} \\ 0 & 0 & 1 \end{pmatrix} \\
k \quad \begin{pmatrix} \frac{(k+n)(g(\text{one}-1)-(i-k-\text{one})(i-j+k+\text{one}))}{(i-k-1)(i-j+k+1)(g-j+k-n+\text{one}+1)} & \frac{g(k+n)(g-j+k+\text{one}+1)(g-j+2\,k+\text{one}+1)}{(i-k-1)(i-j+k+1)(g-j+k-n+\text{one}+1)} & \frac{(i-j-n+1)\,n(i+n-1)(g-j+2\,k+\text{one}+1)}{(i-k-1)(i-j+k+1)(g-j+k-n+\text{one}+1)} \\ \frac{(k+n)(\text{one}-1)(g-j+2\,k+\text{one}+1)}{k(-i+k+1)(i-j+k+1)(g-j+k-n+\text{one}+1)} & \frac{(k+n)(g-j+k+\text{one}+1)(g^2+(-j+2\,k+\text{one}+1)\,g-(i-k-1)(i-j+k+1))}{k(-i+k+1)(i-j+k+1)(g-j+k-n+\text{one}+1)} & \frac{(i-j-n+1)\,n(i+n-1)(g-j+2\,k+\text{one}+1)}{k(-i+k+1)(i-j+k+1)(g-j+k-n+\text{one}+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} -\frac{(g+n)(k+n)(g-j+k-n+\text{one})}{(n+1)(i+n)(-i+j+n)} & 0 & n \\ 0 & -\frac{(g+n)(k+n)(g-j+k-n+\text{one})}{(n+1)(i+n)(-i+j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
h \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

Notice the factors of one-1.

```
replacemats[Factor[% /. one → 1]]
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{i-j-n+1}{i+n} & -\frac{g(2\,i-j+1)\,k\,(g-j+k+1)(i-j-n+1)}{(g+i-j+1)(g-i+k)(i-j+k+1)(i+n)} & \frac{(i-j+1)(2\,i-j+1)(i-j-n+1)\,n}{(g+i-j+1)(-g+i-k)(i-j+k+1)} \\ 0 & \frac{(g-i)(i-k)(g+i-j+k+1)(i-j-n+1)}{(g+i-j+1)(g-i+k)(i-j+k+1)(i+n)} & -\frac{(2\,i-j+1)(i-j-n+1)\,n}{(g+i-j+1)(g-i+k)(i-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
j \quad \begin{pmatrix} -\frac{-g+j-k+n-1}{i-j-n} & \frac{g\,k(g-j+k-n+1)}{(g+i-j+k)(i-j-n)} & \frac{n(i+n-1)}{g+i-j+k} \\ 0 & \frac{(g+i-j)(-i+j-k)(g-j+k-n+1)}{(g-j+k+1)(g+i-j+k)(-i+j+n)} & \frac{n(i+n-1)}{(g-j+k+1)(g+i-j+k)} \\ 0 & 0 & 1 \end{pmatrix} \\
k \quad \begin{pmatrix} -\frac{k+n}{g-j+k-n+2} & \frac{g(g-j+k+2)(g-j+2\,k+2)(k+n)}{(i-k-1)(i-j+k+1)(g-j+k-n+2)} & \frac{(g-j+2\,k+2)(i-j-n+1)\,n(i+n-1)}{(i-k-1)(i-j+k+1)(g-j+k-n+2)} \\ 0 & \frac{(g-i+k+1)(g-j+k+2)(g+i-j+k+1)(k+n)}{k(-i+k+1)(i-j+k+1)(g-j+k-n+2)} & \frac{(g-j+2\,k+2)(i-j-n+1)\,n(i+n-1)}{k(-i+k+1)(i-j+k+1)(g-j+k-n+2)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} -\frac{(g-j+k-n+1)(g+n)(k+n)}{(n+1)(i+n)(-i+j+n)} & 0 & n \\ 0 & -\frac{(g-j+k-n+1)(g+n)(k+n)}{(n+1)(i+n)(-i+j+n)} & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
h \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

```
picheck[%]
```

True

Coordinate *h* died heroically. Matrices k and j are now triangular, too. (As would be g, had we enabled it.) If we repeat the contour exercise we performed for Dixon, we will get the nonterminating generalization of the Pfaff-Säalschütz ${}_3F_2[1]$ valuation. But suppose instead we travel in the (i,n) plane. We'll first need to quell the $O(n^2)$ in the i matrix right column:

```
sidestep[{{#, 0, 0}, {0, #, 0}, {0, 0, 1}} &[(i - j - n + 1) n]][[{1, 4}]] // Factor
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{i-j-n+2}{i+n} & -\frac{g(2\,i-j+1)\,k\,(g-j+k+1)(i-j-n+2)}{(g+i-j+1)(g-i+k)(i-j+k+1)(i+n)} & \frac{(i-j+1)(2\,i-j+1)}{(g+i-j+1)(-g+i-k)(i-j+k+1)} \\ 0 & \frac{(g-i)(i-k)(g+i-j+k+1)(i-j-n+2)}{(g+i-j+1)(g-i+k)(i-j+k+1)(i+n)} & -\frac{2\,i-j+1}{(g+i-j+1)(g-i+k)(i-j+k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} -\frac{(g+n)(-g+j-k+n-1)(k+n)}{(i-j-n+1)\,n(i+n)} & 0 & \frac{1}{i-j-n+1} \\ 0 & -\frac{(g+n)(-g+j-k+n-1)(k+n)}{(i-j-n+1)\,n(i+n)} & -\frac{1}{n(-i+j+n-1)} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

We can get an interesting ${}_pF_q[1]$ by using the i matrix for BOTTOM, but it lacks in its term ratio denominator the factor of i + 1 that we covet for a closed form. This is easily rectified by a coordinate shift

```
recoord[i → i + g + k]
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{g+i-j+k-n+2}{g+i+k+n} & \frac{g\,k(g-j+k+1)(-j+2(g+i+k)+1)(g+i-j+k-n+2)}{i(2\,g+i-j+k+1)(g+i-j+2\,k+1)(g+i+k+n)} & \frac{(g+i-j+k+1)(-j+2(g+i+k)+1)}{i(2\,g+i-j+k+1)(g+i-j+2\,k+1)} \\ 0 & \frac{(g+i)(i+k)(2\,g+i-j+2\,k+1)(g+i-j+k-n+2)}{i(2\,g+i-j+k+1)(g+i-j+2\,k+1)(g+i+k+n)} & \frac{-j+2(g+i+k)+1}{i(2\,g+i-j+k+1)(g+i-j+2\,k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} \frac{(g-j+k-n+1)(g+n)(k+n)}{(g+i-j+k-n+1)\,n(g+i+k+n)} & 0 & \frac{1}{g+i-j+k-n+1} \\ 0 & \frac{(g-j+k-n+1)(g+n)(k+n)}{(g+i-j+k-n+1)\,n(g+i+k+n)} & \frac{1}{(g+i-j+k-n+1)\,n} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

which seemed to gain us i instead of i + 1. But only because the i matrix was insufficiently in ratio form.

```
sidestep[{{1 / i, 0, 0}, {0, 1 / i, 0}, {0, 0, 1}}]
```

$$\left(\begin{array}{l}
i \quad \begin{pmatrix} -\frac{i(g+i-j+k-n+2)}{(i+1)(g+i+k+n)} & \frac{g\,k(g-j+k+1)(-j+2(g+i+k)+1)(g+i-j+k-n+2)}{(i+1)(2\,g+i-j+k+1)(g+i-j+2\,k+1)(g+i+k+n)} & \frac{(g+i-j+k+1)(-j+2(g+i+k)+1)}{(2\,g+i-j+k+1)(g+i-j+2\,k+1)} \\ 0 & \frac{(g+i)(i+k)(2\,g+i-j+2\,k+1)(g+i-j+k-n+2)}{(i+1)(2\,g+i-j+k+1)(g+i-j+2\,k+1)(g+i+k+n)} & \frac{-j+2(g+i+k)+1}{(2\,g+i-j+k+1)(g+i-j+2\,k+1)} \\ 0 & 0 & 1 \end{pmatrix} \\
n \quad \begin{pmatrix} \frac{(g-j+k-n+1)(g+n)(k+n)}{(g+i-j+k-n+1)\,n(g+i+k+n)} & 0 & \frac{i}{g+i-j+k-n+1} \\ 0 & \frac{(g-j+k-n+1)(g+n)(k+n)}{(g+i-j+k-n+1)\,n(g+i+k+n)} & \frac{i}{(g+i-j+k-n+1)\,n} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

```
picheck[%]
```

True

Now we have our i + 1. When does the sum computed by the second row converge?

```
Series[mats[i][[2, 2]], {i, ∞, 1}]
```

$$1-\frac{2\,n}{i}+O\left[\frac{1}{i}\right]^2$$

Oops, that's not quite the true term ratio, which is

```
Series[mats[i][[2, 2]] * DiscreteRatio[mats[i][[2, 3]], i], {i, ∞, 1}]
```

$$1+\frac{-2\,n-1}{i}+O\left(\left(\frac{1}{i}\right)^2\right)$$

Again by comparison with the harmonic series, n can be anywhere east of (0,0):

```
Assuming[Re[n] > 0, MProdi[mats[i], {i, 0, ∞}]] // tim
```

Product::div : Product does not converge. ≫

1.409445,3

$$\left(\begin{array}{l}
0 \quad \begin{pmatrix} 0 & \sum_{\text{k67436}=0}^{\infty} 0 & \sum_{\text{k67434}=0}^{\infty} \frac{(-j+2(g+k+\text{k67434})+1)\sum_{\text{k67438}=0}^{\text{k67434}-1} 0}{(2\,g-j+k+\text{k67434}+1)(g-j+2\,k+\text{k67434}+1)} \\ 0 & 0 & \sum_{\text{k67440}=0}^{\infty} \frac{g\,k(2\,g-j+2\,k+1)(-j+2(g+k+\text{k67440})+1)(g-j+k-n+2)(g+1)_{\text{k67440}-1}(k+1)_{\text{k67440}-1}(2\,g-j+2\,k+2)_{\text{k67440}-1}(g-j+k-n+3)_{\text{k67440}-1}}{(2\,g-j+k+1)(g-j+2\,k+1)(2\,g-j+k+\text{k67440}+1)(g-j+2\,k+\text{k67440}+1)(g+k+n)\,\Gamma(\text{k67440}+1)(2\,g-j+k+2)_{\text{k67440}-1}(g-j+2\,k+2)_{\text{k67440}-1}(g+k+n+1)_{\text{k67440}-1}} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}\right)$$

The top row appears to identically vanish!:

```
Activate[%[[1]]]
```

{0, 0, 0}

But our quarry was the second row.

```
ToPFQ[%%[[2]]]
```

$$\left\{0,\ 0,\ \frac{1}{(2\,g-j+k+1)(g-j+2\,k+1)}(2\,(g+k)-j+1)\ {}_5F_4\left(g,\ \frac{1}{2}(2\,g-j+2\,k+3),\ 2\,g-j+2\,k+1,\ k,\ g-j+k-n+2;\ \frac{1}{2}(2\,g-j+2\,k+1),\ g-j+2\,k+2,\ 2\,g-j+k+2,\ g+k+n;\ 1\right)\right\}$$

So

```
BOTTOM = {%%, %, {0, 0, 1}}
```

The RIGHT matrix is

```
Limit[mats[n], i → ∞] // Factor
```

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{1}{n} \\ 0 & 0 & 1 \end{pmatrix}$$

and the RIGHT product is degenrate:

```
MProdi[%, {n, n0, ∞}]
```

$$\begin{pmatrix} 0 & \sum_{k67444=n_0}^{\infty} 0 & \sum_{k67442=n_0}^{\infty} \frac{\sum_{k67446=n_0}^{k67442-1} 0}{k67442} \\ 0 & 0 & \sum_{k67448=n_0}^{\infty} 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
RIGHT = Activate[%]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When i=0, the LEFT product is merely diagonal:

```
LEFT = MProdi[mats[n] /. i → 0, {n, n, ∞}] // Activate
```

$$\begin{pmatrix} \frac{\Gamma(n)\,\Gamma(g+k+n)}{\Gamma(g+n)\,\Gamma(k+n)} & 0 & 0 \\ 0 & \frac{\Gamma(n)\,\Gamma(g+k+n)}{\Gamma(g+n)\,\Gamma(k+n)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But the TOP matrix is more interesting:

```
Factor[Limit[mats[i], n → ∞]]
```

$$\begin{pmatrix} \frac{i}{i+1} & -\frac{g\,k\,(g-j+k+1)\,(2\,g+2\,i-j+2\,k+1)}{(i+1)\,(2\,g+i-j+k+1)\,(g+i-j+2\,k+1)} & \frac{(g+i-j+k+1)\,(2\,g+2\,i-j+2\,k+1)}{(2\,g+i-j+k+1)\,(g+i-j+2\,k+1)} \\ 0 & -\frac{(g+i)\,(i+k)\,(2\,g+i-j+2\,k+1)}{(i+1)\,(2\,g+i-j+k+1)\,(g+i-j+2\,k+1)} & \frac{2\,g+2\,i-j+2\,k+1}{(2\,g+i-j+k+1)\,(g+i-j+2\,k+1)} \\ 0 & 0 & 1 \end{pmatrix}$$

Due to the i/(i+1), the top row of the product from i=0 is going to compute a vanishing sum (again). When will the middle row sum converge?

```
Series[%[[2, 2]] * DiscreteRatio[%[[2, 3]], i], {i, ∞, 1}]
```

$$-1 + \frac{3-j}{i} + O\left[\frac{1}{i}\right]^2$$

Unlike the harmonic series, this is an alternating sum, so all convergence needs is Re(j)<3.

```
Assuming[Re[j] < 3, MProdi[%%, {i, 0, ∞}]]
```

$$\begin{pmatrix} 0 & \sum_{k1041=0}^{\infty} 0 & \sum_{k1039=0}^{\infty} \frac{(2\,g-j+2\,k+2\,k1039+1)\,\sum_{k6295=0}^{k1039-1} 0}{(2\,g-j+k+k1039+1)\,(g-j+2\,k+k1039+1)} \\ 0 & \prod_{i=0}^{\infty} -\frac{(g+i)\,(i+k)\,(2\,g+i-j+2\,k+1)}{(i+1)\,(2\,g+i-j+k+1)\,(g+i-j+2\,k+1)} & \sum_{k6295=0}^{\infty} \frac{(-1)^{k6295}\,g\,k\,(2\,g-j+2\,k+1)\,(2\,g-j+2\,k+2\,k6295+1)\,(g+1)_{k6295-1}\,(k+1)_{k6295-1}\,(2\,g-j+2\,k+2)_{k6295-1}}{(2\,g-j+k+1)\,(g-j+2\,k+1)\,(2\,g-j+k+k6295+1)\,(g-j+2\,k+k6295+1)\,\Gamma(k6295+1)\,(2\,g-j+k+2)_{k6295-1}\,(g-j+2\,k+2)_{k6295-1}} \\ 0 & 0 & 1 \end{pmatrix}$$

With Re(j)<2, the infinite product will vanish as a bonus:

```
TOP = Assuming[Re[j] < 2, Activate[ToPFQ[MapAt[Activate, %%, 1]]]]
```

```
TOP =
 Assuming[
  Re[j] < 2 &&
   Re[1 + 2 + 2 * g - j + k + 1 / 2 + g - j / 2 + k + 2 + g - j + 2 * k -
       (g + k + 3 / 2 + g - j / 2 + k + 1 + 2 * g - j + 2 * k)] > 0,
  FunctionExpand[Activate[ToPFQ[MapAt[Activate, %, 1]]]]]
```

Product::div : Product does not converge. ≫

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{(2\,g-j+2\,k+1)\,{}_4F_3\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2;-1\right)}{(2\,g-j+k+1)\,(g-j+2\,k+1)} \\ 0 & 0 & 1 \end{pmatrix}$$

So finally,

```
BOTTOM.RIGHT == LEFT.TOP
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{(-j+2\,(g+k)+1)\,{}_5F_4\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1,g-j+k-n+2;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2,g+k+n;1\right)}{(2\,g-j+k+1)\,(g-j+2\,k+1)} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{(2\,g-j+2\,k+1)\,\Gamma(n)\,\Gamma(g+k+n)\,{}_4F_3\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2;-1\right)}{(2\,g-j+k+1)\,(g-j+2\,k+1)\,\Gamma(g+n)\,\Gamma(k+n)} \\ 0 & 0 & 1 \end{pmatrix}$$

and our coveted ${}_5F_4(1)$ is expressed in terms of a ${}_4F_3(-1)$

```
Solve[%[[1, 2, 3]] == %[[2, 2, 3]], Numerator[%[[1, 2, 3]]][[2]]]
```

$$\left\{\left\{{}_5F_4\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1,g-j+k-n+2;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2,g+k+n;1\right) \to \frac{\Gamma(n)\,\Gamma(g+k+n)\,{}_4F_3\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2;-1\right)}{\Gamma(g+n)\,\Gamma(k+n)}\right\}\right\}$$

which is independent of n. So as with the Dixon derivation, choose n to trivialize the ${}_5F_4(1)$:

```
%[[1, 1]] /. n → g - j + k + 2
```

$$1 \to \frac{\Gamma(g-j+k+2)\,\Gamma(2\,g-j+2\,k+2)\,{}_4F_3\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2;-1\right)}{\Gamma(2\,g-j+k+2)\,\Gamma(g-j+2\,k+2)}$$

```
Solve[Equal @@ %, Numerator[%[[2]]][[-1]]]
```

$$\left\{\left\{{}_4F_3\left(g,k,g+k-\frac{j}{2}+\frac{3}{2},2\,g-j+2\,k+1;2\,g-j+k+2,g+k-\frac{j}{2}+\frac{1}{2},g-j+2\,k+2;-1\right) \to \frac{\Gamma(2\,g-j+k+2)\,\Gamma(g-j+2\,k+2)}{\Gamma(g-j+k+2)\,\Gamma(2\,g-j+2\,k+2)}\right\}\right\}$$

which in turn gives us Dougall's ${}_5F_4(1)$:

```
Equal @@ %%%[[1, 1]] /. %[[1, 1]]
```

$${}_5F_4\left(g,k,g-\frac{j}{2}+k+\frac{3}{2},2\,g-j+2\,k+1,g-j+k-n+2;2\,g-j+k+2,g-\frac{j}{2}+k+\frac{1}{2},g-j+2\,k+2,g+k+n;1\right) = \frac{\Gamma(n)\,\Gamma(2\,g-j+k+2)\,\Gamma(g-j+2\,k+2)\,\Gamma(g+k+n)}{\Gamma(g+n)\,\Gamma(k+n)\,\Gamma(g-j+k+2)\,\Gamma(2\,g-j+2\,k+2)}$$

It remains only to canonicalize the parameters:

```
Solve[Thread[%[[1, 1]] == {b, c, a / 2 + 1, a, d}], {g, k, j, n}]
```

```
{{g → b, k → c, j → 1 - a + 2 b + 2 c, n → 1 + a - b - c - d}}
```

```
Simplify[Column[{%%, Equal @@ %%%[[1, 1]]} /. %[[1]]]]
```

$${}_5F_4\left(\frac{a}{2}+1,a,b,c,d;\frac{a}{2},a-b+1,a-c+1,a-d+1;1\right) = \frac{\Gamma(a-b+1)\,\Gamma(a-c+1)\,\Gamma(a-d+1)\,\Gamma(a-b-c-d+1)}{\Gamma(a+1)\,\Gamma(a-b-c+1)\,\Gamma(a-b-d+1)\,\Gamma(a-c-d+1)}$$

$${}_4F_3\left(\frac{a}{2}+1,a,b,c;\frac{a}{2},a-b+1,a-c+1;-1\right) = \frac{\Gamma(a-b+1)\,\Gamma(a-c+1)}{\Gamma(a+1)\,\Gamma(a-b-c+1)}$$

So far we have found closed forms only for some ${}_{p+1}F_p(1)$ and ${}_{p+1}F_p(-1)$. We shall soon sacrifice two coordinates in the Rosetta system to get one for ${}_2F_1(z)$, but there are other ways to evaluate ${}_{p+1}F_p(r)$ for various rational r. One way is through deliberately screwy recoordinatizations, e.g.,

```
{g, h, i, j, k, n} → {h - i + n, k, 1 + h - i - j + k, h + j + n, h, j}
```

```
recoord @@ Thread[%]
```

```
mats[]
```

$$\begin{pmatrix} i & \begin{pmatrix} \frac{-h+i-k}{-h+i-j-n+1} & -\frac{h\,(-h+i-k)\,k\,(-h+i-n+1)}{(-h+i+j)\,(i+j-k)\,(-h+i-j-n+1)} & -\frac{j\,(-h+i-k)\,(-h+i+j-k)\,(h+2\,j+n-1)}{(-h+i+j)\,(i+j-k)\,(-h+i-j-n+1)} \\ 0 & \frac{(i+j)\,(-h+i-k)\,(-h+i-n+1)}{(-h+i+j)\,(i+j-k)\,(-h+i-j-n+1)} & -\frac{j\,(-h+i-k)\,(h+2\,j+n-1)}{(-h+i+j)\,(i+j-k)\,(-h+i-j-n+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} \frac{(h+j)\,(j+k)\,(h-i+j+n)}{4\,(j+1)\left(j+\frac{h+n}{2}\right)\left(j+\frac{1}{2}\,(h+n+1)\right)} & \frac{h\,(h+j)\,k\,(j+k)\,(h-i+n)\,(h-i+j+n)\left(j+\frac{1}{2}\,(i-k+n)\right)}{2\,(j+1)\,(-h+i+j)\,(i+j-k)\,(j-k+n)\left(j+\frac{h+n}{2}\right)\left(j+\frac{1}{2}\,(h+n+1)\right)} & \frac{h^2\,i\,(h-i+k)\left(j+\frac{-3\,j^4+h\,j^3-4\,i\,j^3+4\,k\,j^3-3\,n\,j^3+2\,h^2\,j^2-i^2\,j^2-k^2\,j^2-n^2\,j^2-2\,h\,i\,j^2+3\,h\,k\,j^2+i\,k\,j^2+2\,h\,n\,j^2-3\,i\,n\,j^2+4\,k\,n\,j^2-h\,j^2-2\,h\,k^2\,j+i\,k^2\,j+h\,n^2\,j-i\,n^2\,j+k\,n^2\,j+h^2\,k\,j-i^2\,k\,j+h\,i\,k\,j+h^2\,n\,j-k^2\,n\,j-h\,i\,n\,j+3\,h\,k\,n\,j+i\,k\,n\,j-h^2\,k^2+h\,i\,k^2+h\,k\,n^2-h\,i^2\,k+h^2\,i\,k-h\,k^2\,n+h^2\,k\,n}{h^2\,i} \\ 0 & -\frac{(h+j)\,(i+j)\,(j+k)\,(j+n)\,(h-i+j+n)\,(h+j-k+n)}{4\,(j+1)\,(-h+i+j)\,(i+j-k)\,(j-k+n)\left(j+\frac{h+n}{2}\right)\left(j+\frac{1}{2}\,(h+n+1)\right)} & -\frac{h\,i\,(h-i+k)\left(j+\frac{5\,j^3+3\,h\,j^2+3\,i\,j^2-3\,k\,j^2+6\,n\,j^2+2\,n^2\,j-h\,k\,j-i\,k\,j+2\,h\,n\,j+3\,i\,n\,j-2\,k\,n\,j+i\,n^2+h\,i\,n-i\,k\,n}{2\,h\,i}\right)}{(-h+i+j)\,(i+j-k)\,(j-k+n)\left(j+\frac{h+n}{2}\right)} \\ 0 & 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} \frac{j+k}{h-i+k+1} & -\frac{h\,(j+k)\,(h-i+n)}{(h-i+k+1)\,(-h-j+k-n+1)} & -\frac{j\,(h+2\,j+n-1)}{-h-j+k-n+1} \\ 0 & \frac{(-i-j+k+1)\,(j+k)\,(-j+k-n+1)}{k\,(h-i+k+1)\,(-h-j+k-n+1)} & \frac{j\,(h+2\,j+n-1)}{k\,(-h-j+k-n+1)} \\ 0 & 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{h-i+j+n}{h+2\,j+n} & -\frac{h\,k\,(h-i+j+n)}{(h+2\,j+n)\,(j-k+n)} & -\frac{j\,(h-i+k)}{j-k+n} \\ 0 & \frac{(j+n)\,(h-i+j+n)\,(h+j-k+n)}{(h-i+n)\,(h+2\,j+n)\,(j-k+n)} & \frac{j\,(h-i+k)}{(h-i+n)\,(j-k+n)} \\ 0 & 0 & 1 \end{pmatrix} \\ h & \begin{pmatrix} \frac{(h+j)\,(h-i+j+n)}{(h-i+k+1)\,(h+2\,j+n)} & \frac{(h+j)\,k\,(h-i+j+n)}{(h-i+k+1)\,(h+2\,j+n)} & j \\ 0 & \frac{(h-i-j+1)\,(h+j)\,(h-i+j+n)\,(h+j-k+n)}{h\,(h-i+k+1)\,(h-i+n)\,(h+2\,j+n)} & -\frac{2\,j\left(h+\frac{1}{2}\,(-i+j+n)\right)}{h\,(h-i+n)} \\ 0 & 0 & 1 \end{pmatrix} \\ g & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}$$

```
picheck[%]
```

```
True
```

We gave up g (missing from the rhs of the recoord), and gained triangularity. The j matrix has grown a bit exotic.

```
Series[mats[j][[2, 2]], {j, ∞, 1}]
```

$$-\frac{1}{4} + \frac{-6\,h+4\,i-4\,k-2\,n+3}{8\,j} + O\left(\left(\frac{1}{j}\right)^2\right)$$

I.e., its lower right 2×2 computes a ${}_9F_8(-1/4)$ with four degrees of freedom (h,i,k,n). But it's upper row

```
Series[mats[j][[1, 1]], {j, ∞, 1}]
```

$$\frac{1}{4} + \frac{2\,h-2\,i+2\,k-3}{8\,j} + O\left(\left(\frac{1}{j}\right)^2\right)$$

computes some linear combination of ${}_{p+1}F_p(+1/4)$. I didn't know this was even possible in the same matrix. The system then almost surely separates into two independent sytems of three 2×2s. Since this may get intricate, and ${}_{p+1}F_p(-1/4)$ valuations are uncommon compared to ${}_{p+1}F_p(1/4)$, let's just consider the lower right 2×2s:

```
replacemats[Factor[Take[mats[], 5] /. m_ ?MatrixQ :> Drop[m, 1, 1]]]
```

$$\begin{pmatrix} i & \begin{pmatrix} \frac{(i+j)(-h+i-k)(-h+i-n+1)}{(-h+i+j)(i+j-k)(-h+i-j-n+1)} & -\frac{j(h-i+k)(h+2j+n-1)}{(-h+i+j)(i+j-k)(h-i+j+n-1)} \\ 0 & 1 \end{pmatrix} \\ j & \begin{pmatrix} -\frac{(h+j)(i+j)(j+k)(j+n)(h-i+j+n)(h+j-k+n)}{(j+1)(-h+i+j)(i+j-k)(h+2j+n)(h+2j+n+1)(j-k+n)} & \frac{(h-i+k)(5j^3+3hj^2+3ij^2-3kj^2+6nj^2+2n^2j+2hij-hkj-ikj+2hnj+3inj-2knj+in^2+hin-ikn)}{(h-i-j)(i+j-k)(h+2j+n)(j-k+n)} \\ 0 & 1 \end{pmatrix} \\ k & \begin{pmatrix} \frac{(-i-j+k+1)(j+k)(-j+k-n+1)}{k(h-i+k+1)(-h-j+k-n+1)} & -\frac{j(h+2j+n-1)}{k(h+j-k+n-1)} \\ 0 & 1 \end{pmatrix} \\ n & \begin{pmatrix} \frac{(j+n)(h-i+j+n)(h+j-k+n)}{(h-i+n)(h+2j+n)(j-k+n)} & \frac{j(h-i+k)}{(h-i+n)(j-k+n)} \\ 0 & 1 \end{pmatrix} \\ h & \begin{pmatrix} \frac{(h-i-j+1)(h+j)(h-i+j+n)(h+j-k+n)}{h(h-i+k+1)(h-i+n)(h+2j+n)} & -\frac{j(2h-i+j+n)}{h(h-i+n)} \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

```
picheck[%]
```

True

The j matrix $_{1,1}$ denominator magically has the traditional factor of j+1 so often necessary for closed forms. It virtually guarantees a factor of j in the $_{1,2}$ positions of the other matrices. This will diagonalize those matrices for contour segments departing from j=0, but in a non-j direction.

So BOTTOM will be

```
newindex[MProdi[mats[j], {j, 0, ∞}], j]
```

$$\Big\{\Big\{0, \sum_{j=0}^{\infty} \Big((-1)^j 4^{1-j} h\,i\,k\,(h-i+k)\,n\,(h-i+n)\,(h-k+n)\,\big(2hij+3hj^2+3ij^2+5j^3-hjk-ijk-3j^2k+hin+2hjn+3ijn+6j^2n-ikn-2jkn+in^2+2jn^2\big)\,\text{Pochhammer}[1+h,-1+j]\,\text{Pochhammer}[1+i,-1+j]\,\text{Pochhammer}[1+k,-1+j]\,\text{Pochhammer}[1+n,-1+j]\,\text{Pochhammer}[1+h-i+n,-1+j]\,\text{Pochhammer}[1+h-k+n,-1+j]\Big) \Big/ \Big((h-i)(h-i-j)(i-k)(i+j-k)(k-n)(h+n)(1+h+n)(h+2j+n)(j-k+n)\,\text{Gamma}[1+j]\,\text{Pochhammer}[1-h+i,-1+j]\,\text{Pochhammer}[1+i-k,-1+j]\,\text{Pochhammer}\Big[\tfrac{1}{2}(2+h+n),-1+j\Big]\,\text{Pochhammer}\Big[\tfrac{1}{2}(3+h+n),-1+j\Big]\,\text{Pochhammer}[1-k+n,-1+j]\Big)\Big\}, \{0,1\}\Big\}$$

```
BOTTOM = intopoch[%]
```

$$\begin{pmatrix} 0 & \sum_{j=0}^{\infty} \frac{(-1)^j 2^{1-2j} k(h-i+k)n(h-i+n)(h-k+n)(5j^3+3hj^2+3ij^2-3kj^2+6nj^2+2n^2j+2hij-hkj-ikj+2hnj+3inj-2knj+in^2+hin-ikn)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j} \\ 0 & 1 \end{pmatrix}$$

We can't put this in pFq notation with that big, ugly cubic factor (...+ 5 j^3+ ...). We would need to either force-factor it with messy cubic surds, or specialize two of (h,i,k,n) to get rational factors. But there are hundreds of such specializations, nearly all of which we will need to throw away. Instead, we can re-express the cubic as a linear combination of four small polynomials which combine smoothly with the Pochhammers, breaking the sum into four pFqs. But there are hundreds of ways to do that. Of what use is such a result? Negligible, if it is printed in a table. But in choosing one of these four-term combinations, we have lost no generality, and it is a minor matter of computer algebra to see if any particular pFq is a special case. So it is well worth seeking its closed form.

Let's see if the h marix is suitable for LEFT.

```
monicize[mats[h] /. j → 0, h]
```

$$\Big\{\Big\{\frac{(1+h-i)(h-k+n)}{(1+h-i+k)(h+n)}, 0\Big\}, \{0,1\}\Big\}$$

We need the infinite product of this to converge, meaning it must be 1 + $o(1/h)$. But

```
Series[%[[1, 1]], {h, ∞, 1}]
```

$$1-\frac{2k}{h}+O\Big[\frac{1}{h}\Big]^2$$

which diverges except when k = 0. But for i:

```
monicize[mats[i][[1, 1]] /. j → 0, i]
```

$$\frac{i(-h+i-k)}{(-h+i)(i-k)}$$

```
Series[%, {i, ∞, 1}]
```

$$1+O\Big[\frac{1}{i}\Big]^2$$

Then what about RIGHT?

```
monicize[Limit[mats[i], j → ∞], i]
```

$$\{\{0,0\},\{0,1\}\}$$

Seems almost too good to be true. Will TOP disappoint us?

```
monicize[Limit[mats[j], i → ∞], j]
```

$$\Big\{\Big\{\frac{(h+j)(j+k)(j+n)(h+j-k+n)}{4(1+j)(j-k+n)\left(j+\frac{h+n}{2}\right)\left(j+\frac{1}{2}(1+h+n)\right)}, \frac{h\left(j+\frac{3j^2-jk+hn+3jn-kn+n^2}{2h}\right)}{(j-k+n)\left(j+\frac{h+n}{2}\right)}\Big\}, \{0,1\}\Big\}$$

No, this converges! We're in business.

```
LEFT = MProd[mats[i] /. j → 0, {i, i0, ∞}] /. i0 → i
```

$$\Big\{\Big\{\frac{\text{Gamma}[-h+i]\,\text{Gamma}[i-k]}{\text{Gamma}[i]\,\text{Gamma}[-h+i-k]}, 0\Big\}, \{0,1\}\Big\}$$

```
RIGHT = MProd[{{0, 0}, {0, 1}}, {i, i0, ∞}]
```

$$\{\{0,0\},\{0,1\}\}$$

```
TOP = newindex[MProdi[%%%, {j, 0, ∞}], j]
```

Out[294]=

$$\begin{pmatrix} 0 & \frac{h^2 n(h-k+n)\sum_{j=0}^{\infty}\frac{4^{1-j}(3j^2-kj+3nj+n^2-kn+h(2j+n))(h+1)_{j-1}(k)_j(n+1)_{j-1}(h-k+n+1)_{j-1}}{h\Gamma(j+1)\left(\frac{1}{2}(h+n+3)\right)_{j-1}\left(\frac{1}{2}(h+n+4)\right)_{j-1}(-k+n+1)_j}}{(h+n)(h+n+1)(h+n+2)(n-k)} \\ 0 & 1 \end{pmatrix}$$

This sum, which we'll get in closed form, comes out as three pFq[+1/4] instead of -1/4 !

Now is a good time to convert BOTTOM to pFqs. Grabbing the cubic:

```
cubic = Cases[BOTTOM, _ + 5 * j ^ 3, ∞]
```

$$\{2hij+3hj^2+3ij^2+5j^3-hjk-ijk-3j^2k+hin+2hjn+3ijn+6j^2n-ikn-2jkn+in^2+2jn^2\}$$

let's repeatedly shorten the $(-h+i+3)_{j-2}$ in the denominator:

```
PolynomialQuotientRemainder[%[[1]], j - h + i, j]
```

$$\{8h^2-8hi+2i^2+5j^2-4hk+2ik+8hn-3in-2kn+2n^2+j(8h-2i-3k+6n),\ 8h^3-16h^2i+10hi^2-2i^3-4h^2k+6hik-2i^2k+8h^2n-10hin+3i^2n-2hkn+ikn+2hn^2-in^2\}$$

```
PolynomialQuotientRemainder[%[[1]], j - h + i - 1, j]
```

$$\{5+13h-7i+5j-3k+6n,\ 5+18h+21h^2-12i-28hi+9i^2-3k-7hk+5ik+6n+14hn-9in-2kn+2n^2\}$$

```
PolynomialQuotientRemainder[%[[1]], j - h + i - 2, j]
```

$$\{5, 15+18h-12i-3k+6n\}$$

```
%.{j - h + i - 2, one}
```

$$5(-2-h+i+j)+(15+18h-12i-3k+6n)\,\text{one}$$

```
{%, %%%[[2]]}.{j - h + i - 1, one}
```

$$(5+18h+21h^2-12i-28hi+9i^2-3k-7hk+5ik+6n+14hn-9in-2kn+2n^2)\,\text{one}+(-1-h+i+j)(5(-2-h+i+j)+(15+18h-12i-3k+6n)\,\text{one})$$

```
{%, %%%%%[[2]]}.{j - h + i, one}
```

$$(-h+i+j)\big(\text{one}(21h^2-28hi-7hk+14hn+18h+9i^2+5ik-9in-12i-2kn-3k+2n^2+6n+5)+(-h+i+j-1)(5(-h+i+j-2)+\text{one}(18h-12i-3k+6n+15)))+\text{one}(8h^3-16h^2i-4h^2k+8h^2n+10hi^2+6hik-10hin-2hkn+2hn^2-2i^3-2i^2k+3i^2n+ikn-in^2)$$

Check: is this really the cubic?

```
FullSimplify[cubic[[1]] == % /. one → 1]
```

True

Replacing,

```
BOTTOM /. cubic[[1]] → %%%
```

$$\begin{pmatrix} 0 & \sum_{j=0}^{\infty}\frac{(-1)^j 2^{1-2j}k(h-i+k)n(h-i+n)(h-k+n)\big((8h^3-16ih^2-4kh^2+8nh^2+10i^2h+2n^2h+6ikh-10inh-2knh-2i^3-in^2-2i^2k+3i^2n+ikn)\,\text{one}+(-h+i+j)\big((21h^2-28ih-7kh+14nh+18h+9i^2+2n^2-12i+5ik-3k-9in-2kn+6n+5)\,\text{one}+(-h+i+j-1)(5(-h+i+j-2)+(18h-12i-3k+6n+15)\,\text{one})\big)\big)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j} \\ 0 & 1 \end{pmatrix}$$

Our summand is now in two pieces, multiplied by polynomials in j of degree 0 and 3, the latter containing a factor of j - h + i, which the intopoch function will cancel with the last factor of Pochhammer(3-h+i,-2+j) in the denominator.
Distributing,

```
% /. a_ * (b_ + one * c_) :> a * b + a * c
```

$$\begin{pmatrix} 0 & \sum_{j=0}^{\infty}\left(\frac{(-1)^j 2^{1-2j}k(h-i+k)n(h-i+n)(h-k+n)\big(8h^3-16ih^2-4kh^2+8nh^2+10i^2h+2n^2h+6ikh-10inh-2knh-2i^3-in^2-2i^2k+3i^2n+ikn\big)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j}+\frac{(-1)^j 2^{1-2j}(-h+i+j)k(h-i+k)n(h-i+n)(h-k+n)\big((21h^2-28ih-7kh+14nh+18h+9i^2+2n^2-12i+5ik-3k-9in-2kn+6n+5)\,\text{one}+(-h+i+j-1)(5(-h+i+j-2)+(18h-12i-3k+6n+15)\,\text{one})\big)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j}\right) \\ 0 & 1 \end{pmatrix}$$

Distributing,

```
%715 /. Inactive[Sum][a_ + b_, r_] :> intopoch[Inactive[Sum][a, r] + Inactive[Sum][b, r]]
```

Out[304]=

$$\begin{pmatrix} 0 & \sum_{j=0}^{\infty}\frac{(-1)^{j+1}2^{-2j-1}(h-i+k)(h-i+n)(h-k+n)\big((21h^2+(-28i-7k+14n+18)h+9i^2+2n^2-3k-2ik+6n+i(5k-9n-12)+5)\,\text{one}+(-h+i+j-1)(5(-h+i+j-2)+3(6h-4i-k+2n+5)\,\text{one})\big)(h)_j(i)_j(k)_j(n)_j(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{\Gamma(j+1)(i-h)_{j+1}(i-k)_{j+1}\left(\frac{h+n}{2}\right)_{j+1}\left(\frac{1}{2}(h+n+1)\right)_j(n-k)_{j+1}}+\sum_{j=0}^{\infty}\frac{(-1)^{j+1}2^{-2j-1}(2h-i)(h-i+k)(2h-2i+n)(h-i+n)(h-k+n)(2h-i-k+n)(h)_j(i)_j(k)_j(n)_j(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{\Gamma(j+1)(i-h)_{j+1}(i-k)_{j+1}\left(\frac{h+n}{2}\right)_{j+1}\left(\frac{1}{2}(h+n+1)\right)_j(n-k)_{j+1}} \\ 0 & 1 \end{pmatrix}$$

```
% /. a_ * (b_ + one * c_) :> a * b + a * c
```

$$\begin{pmatrix} 0 & \sum_{j=0}^{\infty}\frac{(-1)^j 2^{1-2j}k(h-i+k)n(h-i+n)(h-k+n)\big(8h^3-16ih^2-4kh^2+8nh^2+10i^2h+2n^2h+6ikh-10inh-2knh-2i^3-in^2-2i^2k+3i^2n+ikn\big)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j}+\sum_{j=0}^{\infty}\left(\frac{(-1)^j 2^{1-2j}k(h-i+k)n(h-i+n)(h-k+n)\big(21h^2-28ih-7kh+14nh+18h+9i^2+2n^2-12i+5ik-3k-9in-2kn+6n+5\big)(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-2}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j}+\frac{(-1)^j 2^{1-2j}(-h+i+j-1)k(h-i+k)n(h-i+n)(h-k+n)(5(-h+i+j-2)+(18h-12i-3k+6n+15)\,\text{one})(h)_j(i)_j(k+1)_{j-1}(n+1)_{j-1}(h-i+n+1)_{j-1}(h-k+n+1)_{j-1}}{(h-i-1)(h-i)(-h+i+2)(i-k)(k-n)(h+n)(h+n+1)\Gamma(j+1)(-h+i+3)_{j-3}(i-k+1)_j\left(\frac{1}{2}(h+n+2)\right)_j\left(\frac{1}{2}(h+n+3)\right)_{j-1}(-k+n+1)_j}\right) \\ 0 & 1 \end{pmatrix}$$

```
▶ % /. Inactive[Sum][a_ + b_, r_] :> intopoch[Inactive[Sum][a, r] + Inactive[Sum][b, r]]
▶In[29]:= %1 /. a_ * (b_ + one * c_) :> a * b + a * c
▼In[30]:= % /. Inactive[Sum][a_ + b_, r_] :> intopoch[Inactive[Sum][a, r] + Inactive[Sum][b, r]]
```

Out[30]=

$$\Big\{\Big\{0,\ \sum_{j=0}^{\infty}\Big(5\,(-1)^j\,2^{1-2j}\,k\,(h-i+k)\,n\,(h-i+n)\,(h-k+n)\,\text{Pochhammer}[h,j]\,\text{Pochhammer}[i,j]\,\text{Pochhammer}[1+k,-1+j]\,\text{Pochhammer}[1+n,-1+j]\,\text{Pochhammer}[1+h-i+n,-1+j]\,\text{Pochhammer}[1+h-k+n,-1+j]\Big)\Big/\Big((-1+h-i)\,(h-i)\,(2-h+i)\,(i-k)\,(k-n)\,(h+n)\,(1+h+n)\,\text{Gamma}[1+j]\,\text{Pochhammer}[3-h+i,-5+j]\,\text{Pochhammer}[1+i-k,j]\,\text{Pochhammer}\Big[\tfrac{1}{2}(2+h+n),j\Big]\,\text{Pochhammer}\Big[\tfrac{1}{2}(3+h+n),-1+j\Big]\,\text{Pochhammer}[1-k+n,j]\Big)+$$

$$\sum_{j=0}^{\infty}\Big((-1)^j\,2^{1-2j}\,k\,(h-i+k)\,n\,(h-i+n)\,(h-k+n)\,(15+18h-12i-3k+6n)\,\text{Pochhammer}[h,j]\,\text{Pochhammer}[i,j]\,\text{Pochhammer}[1+k,-1+j]\,\text{Pochhammer}[1+n,-1+j]\,\text{Pochhammer}[1+h-i+n,-1+j]\,\text{Pochhammer}[1+h-k+n,-1+j]\Big)\Big/\Big((-1+h-i)\,(h-i)\,(2-h+i)\,(i-k)\,(k-n)\,(h+n)\,(1+h+n)\,\text{Gamma}[1+j]\,\text{Pochhammer}[3-h+i,-4+j]\,\text{Pochhammer}[1+i-k,j]\,\text{Pochhammer}\Big[\tfrac{1}{2}(2+h+n),j\Big]\,\text{Pochhammer}\Big[\tfrac{1}{2}(3+h+n),-1+j\Big]\,\text{Pochhammer}[1-k+n,j]\Big)+$$

$$\sum_{j=0}^{\infty}\Big((-1)^j\,2^{1-2j}\,k\,(h-i+k)\,n\,(h-i+n)\,(h-k+n)\,\big(5+18h+21h^2-12i-28hi+9i^2-3k-7hk+5ik+6n+14hn-9in-2kn+2n^2\big)\,\text{Pochhammer}[h,j]\,\text{Pochhammer}[i,j]\,\text{Pochhammer}[1+k,-1+j]\,\text{Pochhammer}[1+n,-1+j]\,\text{Pochhammer}[1+h-i+n,-1+j]\,\text{Pochhammer}[1+h-k+n,-1+j]\Big)\Big/\Big((-1+h-i)\,(h-i)\,(2-h+i)\,(i-k)\,(k-n)\,(h+n)\,(1+h+n)\,\text{Gamma}[1+j]\,\text{Pochhammer}[3-h+i,-3+j]\,\text{Pochhammer}[1+i-k,j]\,\text{Pochhammer}\Big[\tfrac{1}{2}(2+h+n),j\Big]\,\text{Pochhammer}\Big[\tfrac{1}{2}(3+h+n),-1+j\Big]\,\text{Pochhammer}[1-k+n,j]\Big)+$$

$$\sum_{j=0}^{\infty}\Big((-1)^j\,2^{1-2j}\,k\,(h-i+k)\,n\,(h-i+n)\,(h-k+n)\,\big(8h^3-16h^2i+10hi^2-2i^3-4h^2k+6hik-2i^2k+8h^2n-10hin+3i^2n-2hkn+ikn+2hn^2-in^2\big)\,\text{Pochhammer}[h,j]\,\text{Pochhammer}[i,j]\,\text{Pochhammer}[1+k,-1+j]\,\text{Pochhammer}[1+n,-1+j]\,\text{Pochhammer}[1+h-i+n,-1+j]\,\text{Pochhammer}[1+h-k+n,-1+j]\Big)\Big/\Big((-1+h-i)\,(h-i)\,(2-h+i)\,(i-k)\,(k-n)\,(h+n)\,(1+h+n)\,\text{Gamma}[1+j]\,\text{Pochhammer}[3-h+i,-2+j]\,\text{Pochhammer}[1+i-k,j]\,\text{Pochhammer}\Big[\tfrac{1}{2}(2+h+n),j\Big]\,\text{Pochhammer}\Big[\tfrac{1}{2}(3+h+n),-1+j\Big]\,\text{Pochhammer}[1-k+n,j]\Big)\Big\},\ \{0,1\}\Big\}$$

At last, there are no naked j polynomials in the summands, and we can convert to a linear combination of four ${}_6F_5$:

```
▶In[354]:= ToPFQ[%30].RIGHT == LEFT.TOP
```

Equating the upper left elements

```
▼In[355]:= #[[1, 2]] & /@ %
```

Out[355]=

$$\frac{1}{(h+n)(i-k)(k-n)}(h-i+k)\big(21h^2+h(-28i-7k+14n+18)+9i^2+i(5k-9n-12)-2kn-3k+2n^2+6n+5\big)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{i-h,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+$$
$$\frac{1}{(h+n)(i-k)(k-n)}5(h-i+1)(h-i+2)(h-i+k)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-2,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-\frac{1}{(h+n)(i-k)(k-n)}3(h-i+1)(h-i+k)(6h-4i-k+2n+5)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-$$
$$\frac{1}{(h-i)(h+n)(i-k)(k-n)}(2h-i)(h-i+k)(2h-2i+n)(2h-i-k+n)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i+1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)=$$
$$\frac{\Gamma(i-h)\,\Gamma(i-k)\sum_{j=0}^{\infty}\frac{4^{2-j}h^2(h-k+n)\left(j+\frac{3j^2-kj+3nj+n^2+hn-kn}{2h}\right)(h+1)_{j-1}(k)_j(n)_j(h-k+n+1)_{j-1}}{(h+n)(h+n+1)(h+n+2)(h+n+3)(n-k)\Gamma(j+1)(-k+n+1)_j\left(\frac{1}{2}(h+n+2)+1\right)_{j-1}\left(\frac{1}{2}(h+n+3)+1\right)_{j-2}}}{\Gamma(i)\,\Gamma(-h+i-k)}$$

A quick check: $i = 0$ will terminate all four ${}_6F_5$ after one term:

```
▼In[37]:= #[[1, 2]] & /@ Activate[%%] /. i → 0
```

Out[37]=

$$-\frac{5(1+h)(2+h)(h+k)}{k(k-n)(h+n)}+\frac{2(h+k)(2h+n)(2h-k+n)}{k(k-n)(h+n)}+\frac{3(1+h)(h+k)(5+6h-k+2n)}{k(k-n)(h+n)}-\frac{(h+k)\big(5+21h^2-3k+6n-2kn+2n^2+h(18-7k+14n)\big)}{k(k-n)(h+n)}==0$$

```
▼In[38]:= FullSimplify[%]
Out[38]= True
```

Aah! On second thought, gaah! Zero?? The righthand side is 0? The TOP sum got divided by Gamma(i) ! Then i = -1, -2, ... won't work, either. Our only hope for an *i* value that terminates BOTTOM without annihilating TOP is

```
▼In[249]:= MapAt[Activate, %35 /. i → h + n, {1, 1, 2}]
```

Out[249]=

$$\begin{pmatrix}0 & \frac{5(1-n)(2-n)}{(h+n)(h-k+n)}+\frac{(h-k)(h-n)(2h+n-2(h+n))}{n(h+n)(h-k+n)}+\frac{21h^2+(-7k+14n-28(h+n)+18)h+2n^2+9(h+n)^2-3k-2kn+6n+(h+n)(5k-9n-12)+5}{(h+n)(h-k+n)}-\frac{3(1-n)(6h-k+2n-4(h+n)+5)}{(h+n)(h-k+n)}\\ 0 & 1\end{pmatrix}=$$
$$\begin{pmatrix}0 & \frac{\Gamma(n)\,\Gamma(h-k+n)\sum_{j=0}^{\infty}\frac{4^{2-j}h^2(h-k+n)\left(j+\frac{3j^2-kj+3nj+n^2+hn-kn}{2h}\right)(h+1)_{j-1}(k)_j(n)_j(h-k+n+1)_{j-1}}{(h+n)(h+n+1)(h+n+2)(h+n+3)(n-k)\Gamma(j+1)(-k+n+1)_j\left(\frac{1}{2}(h+n+2)+1\right)_{j-1}\left(\frac{1}{2}(h+n+3)+1\right)_{j-2}}}{\Gamma(h+n)\,\Gamma(n-k)}\\ 0 & 1\end{pmatrix}$$

```
▼In[334]:= intopoch[%311]
```

Out[334]=

$$1=\sum_{j=0}^{\infty}\frac{\Gamma(n)\,(h)_j\,(k)_j\,(n)_j\big(n(h+3j-k)+j(2h+3j-k)+n^2\big)\,\Gamma(h-k+n+1)\,(h-k+n+1)_{j-1}}{\Gamma(j+1)\,\Gamma(h+2j+n+1)\,\Gamma(j-k+n+1)}$$

"Random" numerical test:

```
▼In[335]:= % /. h → 1 / 3 /. k → .2 /. n → 3 / 7 // Activate
Out[335]= True
```

We could (and at some point should) break the TOP sum into three ${}_4F_3[1/4]$ (or two ${}_5F_4$) with three freedoms, but for now we've simply evaluated it in one piece by exploiting its independence of the *i* parameter, which we then chose to terminate the BOTTOM sum(s). We can now express BOTTOM by cancelling above rendition of TOP (which conveniently is identically 1) against the relation of TOP to BOTTOM we got from the contour:

```
▼In[340]:= MapThread[Divide, {#[[1, 2]] & /@ List @@ %35, List @@ %334}]
```

Out[340]=

$$\Big\{\frac{1}{(h+n)(i-k)(k-n)}(h-i+k)\big(21h^2+h(-28i-7k+14n+18)+9i^2+i(5k-9n-12)-2kn-3k+2n^2+6n+5\big)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{i-h,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+$$
$$\frac{1}{(h+n)(i-k)(k-n)}5(h-i+1)(h-i+2)(h-i+k)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-2,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-\frac{1}{(h+n)(i-k)(k-n)}3(h-i+1)(h-i+k)(6h-4i-k+2n+5)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-$$
$$\frac{1}{(h-i)(h+n)(i-k)(k-n)}(2h-i)(h-i+k)(2h-2i+n)(2h-i-k+n)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i+1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big),$$
$$\frac{\Gamma(i-h)\,\Gamma(i-k)\sum_{j=0}^{\infty}\frac{4^{2-j}h^2(h-k+n)\left(j+\frac{3j^2-kj+3nj+n^2+hn-kn}{2h}\right)(h+1)_{j-1}(k)_j(n)_j(h-k+n+1)_{j-1}}{(h+n)(h+n+1)(h+n+2)(h+n+3)(n-k)\Gamma(j+1)(-k+n+1)_j\left(\frac{1}{2}(h+n+2)+1\right)_{j-1}\left(\frac{1}{2}(h+n+3)+1\right)_{j-2}}}{\Gamma(i)\,\Gamma(-h+i-k)\sum_{j=0}^{\infty}\frac{\big(n^2+(h+3j-k)n+j(2h+3j-k)\big)\,\Gamma(n)\,\Gamma(h-k+n+1)\,(h)_j(k)_j(n)_j(h-k+n+1)_{j-1}}{\Gamma(j+1)\,\Gamma(h+2j+n+1)\,\Gamma(j-k+n+1)}}\Big\}$$

But when we try to cancel the explicit summations,

```
▼In[341]:= Equal @@ FullSimplify@outofsum@%
```

Out[341]=

$$\frac{1}{(h+n)(i-k)(k-n)}(h-i+k)\Big(\big(21h^2+h(-28i-7k+14n+18)+9i^2+i(5k-9n-12)+2n(n-k)-3k+6n+5\big)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{i-h,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+5(h-i+1)(h-i+2)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-2,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-3(h-i+1)(6h-4i-k+2n+5)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+\frac{1}{i-h}(2h-i)(2h-2i+n)(2h-i-k+n)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i+1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)\Big)=$$
$$\frac{h^2\,\Gamma(i-h)\,(h-k+n)\,\Gamma(i-k)\sum_{j=0}^{\infty}\frac{2^{3-2j}\big(n^2+(h+3j-k)n+j(2h+3j-k)\big)(h+1)_{j-1}(k)_j(n)_j(h-k+n+1)_{j-1}}{h\,\Gamma(j+1)\left(\frac{1}{2}(h+n+4)\right)_{j-1}\left(\frac{1}{2}(h+n+5)\right)_{j-2}(-k+n+1)_j}}{(h+n)(h+n+1)(h+n+2)(h+n+3)\,\Gamma(i)\,(n-k)\,\Gamma(n)\,\Gamma(-h+i-k)\,\Gamma(h-k+n+1)\sum_{j=0}^{\infty}\frac{\big(n^2+(h+3j-k)n+j(2h+3j-k)\big)(h)_j(k)_j(n)_j(h-k+n+1)_{j-1}}{\Gamma(j+1)\,\Gamma(h+2j+n+1)\,\Gamma(j-k+n+1)}}$$

They have grown so formally dissimilar that *Mathematica* can't canonicalize them into canceling. No problem! Whatever is their ratio is also the ratio of their individual terms and partial sums! Namely

```
▼In[342]:= FullSimplify[Activate[Table[%[[2]], {∞, 0, 3}]]]
```

Out[342]=

$$\Big\{\frac{\Gamma(i-h)\,\Gamma(h+n)\,\Gamma(i-k)\,\Gamma(n-k)}{\Gamma(i)\,\Gamma(n)\,\Gamma(-h+i-k)\,\Gamma(h-k+n)},\ \frac{\Gamma(i-h)\,\Gamma(h+n)\,\Gamma(i-k)\,\Gamma(n-k)}{\Gamma(i)\,\Gamma(n)\,\Gamma(-h+i-k)\,\Gamma(h-k+n)},\ \frac{\Gamma(i-h)\,\Gamma(h+n)\,\Gamma(i-k)\,\Gamma(n-k)}{\Gamma(i)\,\Gamma(n)\,\Gamma(-h+i-k)\,\Gamma(h-k+n)},\ \frac{\Gamma(i-h)\,\Gamma(h+n)\,\Gamma(i-k)\,\Gamma(n-k)}{\Gamma(i)\,\Gamma(n)\,\Gamma(-h+i-k)\,\Gamma(h-k+n)}\Big\}$$

(repeated four times for emphasis).

```
▶In[343]:= %341[[1]] /. a_ * s : (_ * Inactive[HypergeometricPFQ][__] + _) :> s == %342[[1]] / a
```

Here, finally, is our "quadruple ${}_6F_5[-1/4]$ identity:

```
▼In[344]:= MapAt[FullSimplify, %, 2]
```

Out[344]=

$$\big(21h^2+h(-28i-7k+14n+18)+9i^2+i(5k-9n-12)+2n(n-k)-3k+6n+5\big)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{i-h,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+5(h-i+1)(h-i+2)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-2,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)-3(h-i+1)(6h-4i-k+2n+5)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i-1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)+\frac{1}{i-h}(2h-i)(2h-2i+n)(2h-i-k+n)\,\text{HypergeometricPFQ}\Big(\{h,i,k,h-i+n,h-k+n,n\},\Big\{-h+i+1,i-k+1,\tfrac{1}{2}(h+n+2),\tfrac{1}{2}(h+n+1),-k+n+1\Big\},-\tfrac{1}{4}\Big)=\frac{\Gamma(i-h)\,\Gamma(h+n+1)\,\Gamma(i-k+1)\,\Gamma(-k+n+1)}{\Gamma(i)\,\Gamma(n)\,\Gamma(-h+i-k+1)\,\Gamma(h-k+n)}$$

Test it for the other five instant terminations:

```
▼In[345]:= FullSimplify /@ Activate[% /. h → 0]
Out[345]= True
▼In[346]:= FullSimplify /@ Activate[%% /. k → 0]
```

Out[346]=

$$-\frac{i\,n\,(h+n)}{h-i}==\frac{i\,n\,(h+n)}{-h+i}$$

```
▼In[347]:= FullSimplify /@ Activate[%%% /. i → h + n]
Out[347]= True
▼In[348]:= FullSimplify /@ Activate[%%%% /. k → h + n]
Out[348]= True
▼In[349]:= FullSimplify /@ Activate[%%%%% /. n → 0]
Out[349]= True
```

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