Measured bounce times :

In[300]:= {75, 233, 703, 2109, 6323, 18 961, 56 871, 170 597}

Out[300]= {75, 233, 703, 2109, 6323, 18 961, 56 871, 170 597}

In[301]:= **Table[%300[[i + 1]] - %300[[i]] * 3, {i, 1, 7}]**

Out[301]= {8, 4, 0, -4, -8, -12, -16}

In[302]:= **RSolve[{a[n] == 3 a[n - 1] - 4 * (n - 3), a[1] == 75}, a[n], n]**

Out[302]= $\left\{\left\{a[n] \to \frac{1}{3}\left(-9 + 76 \times 3^{n} + 6\,n\right)\right\}\right\}$

Measured beginnings of glider periods :

In[303]:= **{1, 27, 261, 2367}**

Out[303]= {1, 27, 261, 2367}

In[304]:= **RSolve[{a[1] == 1, a[n] == 9 a[n - 1] + 18}, a[n], n]**

Out[304]= $\left\{\left\{a[n] \to \frac{1}{36}\left(-81 + 13 \times 9^{n}\right)\right\}\right\}$

Here are the glider periods : (0 <= k < ∞)

In[305]:= $\textbf{Table}\left[\frac{1}{36}\left(-81 + 13 \times 9^{n}\right) + 3^{2\,n+1}\,k,\ \{n, 1, 10\}\right]$

Out[305]= {1 + 27 k, 27 + 243 k, 261 + 2187 k, 2367 + 19 683 k, 21 321 + 177 147 k,
191 907 + 1 594 323 k, 1 727 181 + 14 348 907 k, 15 544 647 + 129 140 163 k,
139 901 841 + 1 162 261 467 k, 1 259 116 587 + 10 460 353 203 k}

For a glider to escape through the wall, two of these (each with their own k) must coincide. Here' s a proof that no two of these sequences have any values in common :

(*Generally, to for two sequences (a and b, with a<b) to coincide,*)

$$\frac{1}{36}\left(-81 + 13 \times 9^{a}\right) + 3^{2\,a+1}\,k == \frac{1}{36}\left(-81 + 13 \times 9^{b}\right) \left(\mathrm{mod}\ 3^{2\,b+1}\right)$$

$$3^{2\,a+1}\,k == \frac{1}{36}\left(-81 + 13 \times 9^{b}\right) - \frac{1}{36}\left(-81 + 13 \times 9^{a}\right) \left(\mathrm{mod}\ 3^{2\,b+1}\right)$$

$$3^{2\,a+1}\,k == \frac{1}{36}\left(-81 + 13 \times 9^{b} + 81 - 13 \times 9^{a}\right) \left(\mathrm{mod}\ 3^{2\,b+1}\right)$$

$$3^{2\,a+1}\,k == \frac{13}{36}\left(9^{b} - 9^{a}\right) \left(\mathrm{mod}\ 3^{2\,b+1}\right)$$

(*Since $3^{2a+1}\backslash 3^{2b+1}$, this implies that *)

$$0 == \frac{13}{36}\left(9^{b} - 9^{a}\right) \left(\mathrm{mod}\ 3 * 9^{a}\right)$$

**(* Both 13 and 4 have unique inverses mod $3*9^{a}$,
since they are both coprime to $3*9^{a}$. We then divide and
multiply both sides by these inverses, respectively, to get*)**

$$0 == \frac{(9^b - 9^a)}{9} \ (\text{mod } 3 * 9^a)$$

$$0 == 9^{b-1} - 9^{a-1} \ (\text{mod } 3 * 9^a)$$

```
(* Switching back to "normal" arithmetic, *)
```

$$9^{b-1} - 9^{a-1} + r * 3 * 9^a == 0$$

$$9^{b-a} - 1 + 27\, r == 0$$

$$9 \left(9^{b-a-1} + 3\, r\right) == 1$$

$$\left(9^{b-a-1} + 3\, r\right) == \frac{1}{9}$$

Since b - a - 1 >= 0 and r is an integer, the l.h.s. is an integer, while the r.h.s. is a fraction. By reductio ad absurdum, two sequences cannot coincide, and thus no glider ever makes it out through the upper line.

Measured glider positions on lower line :

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{12, 90, 792, 7110}
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In[333]:=
```
RSolve[{a[1] == 12, a[n] == 9 a[n - 1] - 18}, a[n], n]
```

Out[333]= $\left\{\left\{a[n] \to \frac{1}{12} \left(27 + 13 \times 3^{2\, n}\right)\right\}\right\}$

Going along the same lines as before,

$$\frac{1}{12} (27 + 13 \times 9^a) + 3^{2\, a}\, k == \frac{1}{12} \left(27 + 13 \times 9^b\right) \left(\text{mod } 3^{2\, b}\right)$$

$$3^{2\, a}\, k == \frac{13}{12} \left(9^b - 9^a\right) \left(\text{mod } 3^{2\, b}\right)$$

```
(*implies that*)
```

$$0 == \frac{13}{12} \left(9^b - 9^a\right) \ (\text{mod } 9^a)$$

```
(*By the unique-inverse trick, *)
```

$$0 == 3^{2\, b-1} - 3^{2\, a-1} \left(\text{mod } 3^{2\, a}\right)$$

$$3^{2\, b-1} - 3^{2\, a-1} + 3^{2\, a}\, r == 0$$

$$3^{2\, b-2\, a} - 1 + 3\, r == 0$$

$$3 \left(3^{2\, b-2\, a-1} + r\right) == 1$$

and thus, again by reductio ad absurdum, gliders cannot escape through the bottom line. Therefore, gliders can't escape. ■