

Lisp + Calculus = Identities

R. Wm. Gosper

Symbolics, Inc.

700 East El Camino Real, Suite 120, Mountain View, California 94040

$$\sum_{k>-\infty} \frac{\prod_{n \geq 2} 1 + \tan \frac{2k+1}{(-2)^n} \pi}{(2k+1)^2} e^{i\pi \frac{2k+1}{260}} = \frac{2\pi^2}{9}, \quad (1)$$

$$\sum_{k>-\infty} \frac{\prod_{n \geq 2} 1 + \tan \frac{2k+1}{(-2)^n} \pi}{(2k+1)^2} e^{i\pi \frac{2k+1}{261}} = \frac{2\pi^2}{9} + \frac{4474421120 + 52016651039i}{2^{42} + i} \frac{5\pi^2}{36}, \quad (2)$$

$$\sum_{k>-\infty} \frac{1 - \cos(\pi(k+\phi)t) \prod_{n \geq 1} 1 + s \left(\sin \frac{k+\phi}{3^n} \pi \right)^2}{k+\phi} = 0, \quad -1 < t < 1, \quad (3)$$

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}} \prod_{k \geq 0} 1 - q^{k-n} - ap^k + cp^k q^k}{a - cq^n (q-1)(q^2-1) \dots (q^n-1)} = \frac{\left(1 - \prod_{n \geq 0} 1 - ap^n\right) \prod_{n \geq 1} 1 - q^n}{-a}, \quad (4)$$

$$\sum_{n \geq 0} \frac{(-)^n \cos \pi \sqrt{\left(n + \frac{1}{2}\right)^2 + a^2}}{\left(n + \frac{1}{2}\right) \left(n + \frac{1}{2} - a\right) \left(n + \frac{1}{2} + a\right)} = \frac{\pi}{2} \frac{\sin^2 \pi a - 2 \sin^2 \frac{\pi a}{\sqrt{2}}}{a^2 \cos \pi a}. \quad (5)$$

(Identity (3) was derived assuming ϕ real and $|s+2| < 2$, but appears to hold more widely. Identity (4) requires $|p|, |q| < 1$, and these suffice.)

In fathering Lisp, John McCarthy grandfathered computer algebra, all the while maintaining a clandestine affair with mathematical analysis. Thus, he must have been interested to hear that analysis and computer algebra were shackled up at my place. Sadly, all that issued was a bunch of useless identities—proverbial bicycles for John’s proverbial fish.

Yet can we be sure that fish do not find bicycles *entertaining*? To keep the dosage small, we have selected five sums of infinite products. (The $\cos \sqrt{}$ in (5) comes from an infinite product.)

This selection of identities is a little misleading, since the first three came from Fourier analysis of fractal curves, while the last two came from a (so far) unrelated series derangement scheme. Perhaps the juxtaposition will help someone find a connection.

1 Rotational Symmetry

Let $z(t)$, $-1 \leq t \leq 1$, be an arc in the complex plane, and $z_m(t)$ be the period $2m$ arrangement of m such arcs around a regular m -gon, so that $z_m(t+2) = e^{2i\pi/m} z_m(t)$. Suppose further that z_m is nice enough to equal its Fourier approximation,

$$z_m(t) = \sum_j a_j e^{i\pi j t/m}.$$

Then the Fourier coefficients are:

$$\begin{aligned} a_j &= \frac{1}{2m} \int_{-m}^m e^{-i\pi j t/m} z_m(t) dt = \frac{1}{2m} \int_{-1}^{2m-1} e^{-i\pi j t/m} z_m(t) dt \\ &= \frac{1}{2m} \sum_{n=0}^{m-1} \int_{-1}^1 e^{-i\pi j(t+2n)/m} z_m(t+2n) dt \\ &= \frac{1}{2m} \sum_{n=0}^{m-1} e^{-i\pi(j-1)2n/m} \int_{-1}^1 e^{-i\pi j t/m} z_m(t) dt \\ &= \begin{cases} \frac{1}{2} \int_{-1}^1 e^{-i\pi j t/m} z_m(t) dt, & \text{if } m|j-1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let us further stipulate that the arcs remain congruent as we vary m , and each of the $z_m(t)$ is rotated so as to differ from $z(t)$ by only a constant $c_m := c \cot(\pi/m) = c \cot \pi(k+1/m)$, for $-1 \leq t \leq 1$. (For fixed side-length, m -gons are proportional to $\cot(\pi/m)$.) Then we can write:

$$\begin{aligned}
z_m(t) &= \sum_k a_{km+1} e^{i\pi(k+1/m)t} = \sum_k A(k+1/m) e^{i\pi(k+1/m)t}, \\
A(x) &:= \frac{1}{2} \int_{-1}^1 e^{-i\pi xt} (z(t) + c \cot(\pi x)) dt \\
&=: b(x) + c \frac{\cos \pi x}{\pi x}.
\end{aligned}$$

Thus, if we can compute the function $b(x) := \int_{-1}^1 e^{-i\pi xt} z(t) dt / 2$, we know the Fourier series that arranges the arcs around an arbitrary m -gon. This is actually rather magic, and is not equivalent to taking the m th root of some $z_1(t)$, which would distort the arcs, or to taking every m th term of some series free of m . In fact, m can be fractional, which arranges the arcs around a “star”.

2 A Fractal Arc

The arc $z(t)$ I wish to repeat around the m -gon is continuous, and joins $-i$ to i , so that $c = 1$ and $z_m([-1, 1]) = \cot \pi/m + z([-1, 1])$, and the m -gon has side length 2. This will make z_m continuous, but we will give it unbounded variation, just to annoy the lawyers.

The arc is to be dyadically self-similar: the segment joining $z(-1) = -i$ to the “midpoint” $s := z(0)$ will be $z(t)$ shifted and scaled so that $-i$ maps to s and i maps to $-i$. Symmetrically, the segment from s to i will mimic $z(t)$ from i to $-i$.

Koch’s snowflake curve is the case $m = 3, s = 1/\sqrt{3}$, or alternatively, $m = 6, s = -1/\sqrt{3}$, of $z_m(t)$ [Man, pp. 42–44]. Cesàro’s squarefilling function is $m = 2, s = 1$, or $m = 4, s = -1$ [Man, p. 57, 64–65].

For simplicity, temporarily assume s real, and remove the quotes from “midpoint”. Then (using homogeneous coordinates),

$$\begin{pmatrix} z(t) \\ 1 \end{pmatrix} = \begin{cases} M \begin{pmatrix} z(-1-2t) \\ 1 \end{pmatrix}, & -1 \leq t \leq 0; \\ \overline{M} \begin{pmatrix} z(1-2t) \\ 1 \end{pmatrix}, & 0 \leq t \leq 1; \end{cases}$$

where

$$M := \begin{pmatrix} \frac{is-1}{2} & \frac{s-i}{2} \\ 0 & 1 \end{pmatrix}.$$

Thus, the “balanced negabinary” expansion of t defines $z(t)$ as an infinite product of M and $\overline{M}$, and this product converges to

$$\begin{pmatrix} 0 & z(t) \\ 0 & 1 \end{pmatrix}$$

for $\max\{|s - i|, |s + i|\} < 2$, sufficing to recursively define $z(t)$, without need of termination conditions, such as $z(0) = s$ or $z(1) = i$. Furthermore, for rational t , where the negabinary expansion repeats, we can get exact valuations of z :

$$\begin{pmatrix} z(1/3) \\ 1 \end{pmatrix} = \overline{M} \begin{pmatrix} z(1/3) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-is-1}{2}z(1/3) + \frac{s+i}{2} \\ 1 \end{pmatrix},$$

that is,

$$z(1/3) = \frac{s+i}{is+3}.$$

Similarly,

$$\begin{aligned} \begin{pmatrix} z(0) \\ 1 \end{pmatrix} &= M \begin{pmatrix} z(-1) \\ 1 \end{pmatrix} = MM \begin{pmatrix} z(1) \\ 1 \end{pmatrix} = MM\overline{M} \begin{pmatrix} z(-1) \\ 1 \end{pmatrix} \\ &= M \begin{pmatrix} \frac{s^2+1}{4}z(-1) + \frac{s^2-3}{4}i \\ 1 \end{pmatrix}, \end{aligned}$$

so that $z(-1) = -i$ and $z(0) = s$, as expected. Thus, we have the balanced negabinary expansions

$$\begin{aligned} 1/3 &= (. \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2} = (1. \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2}, \\ 0 &= (. 1 \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2} = (. \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2} \\ &= (1. \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2} = (\overline{1}. \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \dots)_{-2} \\ &\vdots \end{aligned}$$

where $\overline{1}$ means the digit -1 . So this system has infinitely many representations of every number, none of which terminates, for lack of a 0 digit. (But then, what do they do on the left?). Whoever named this “balanced” must have subcontracted on the Tower of Pisa.

3 The Arc’s Fourier Series

Define $\omega := e^{-i\pi x}$ and consider the sequence

$$\begin{aligned}
\begin{pmatrix} z(0) \\ 1 \end{pmatrix} &= \begin{pmatrix} z(0) \\ 1 \end{pmatrix} \\
\frac{\omega^{-1/2}M + \omega^{1/2}\overline{M}}{2} \begin{pmatrix} z(0) \\ 1 \end{pmatrix} &= \begin{pmatrix} \frac{\omega^{-1/2}z(-\frac{1}{2}) + \omega^{1/2}z(\frac{1}{2})}{2} \\ \cos(-\pi x/2) \end{pmatrix} \\
\frac{\omega^{-1/2}M + \omega^{1/2}\overline{M}}{2} \frac{\omega^{1/4}M + \omega^{-1/4}\overline{M}}{2} \begin{pmatrix} z(0) \\ 1 \end{pmatrix} &= \\
&\begin{pmatrix} \frac{\omega^{-3/4}z(-\frac{3}{4}) + \omega^{-1/4}z(-\frac{1}{4}) + \omega^{1/4}z(\frac{1}{4}) + \omega^{3/4}z(\frac{3}{4})}{4} \\ \cos(-\pi x/2) \cos(\pi x/4) \end{pmatrix} \\
&\vdots
\end{aligned}$$

In the limit, the sum on the right becomes an integral:

$$\begin{aligned}
&\begin{pmatrix} 0 & \frac{1}{2} \int_{-1}^1 \omega^t z(t) dt \\ 0 & d(x) \end{pmatrix} \\
&= \frac{\omega^{-1/2}M + \omega^{1/2}\overline{M}}{2} \frac{\omega^{1/4}M + \omega^{-1/4}\overline{M}}{2} \frac{\omega^{-1/8}M + \omega^{1/8}\overline{M}}{2} \dots \\
&= \frac{\omega^{-1/2}M + \overline{\omega^{-1/2}M}}{2} \frac{\omega^{1/4}M + \overline{\omega^{1/4}M}}{2} \frac{\omega^{-1/8}M + \overline{\omega^{-1/8}M}}{2} \dots \\
&= \Re(\omega^{-1/2}M) \Re(\omega^{1/4}M) \Re(\omega^{-1/8}M) \Re(\omega^{1/16}M) \dots,
\end{aligned}$$

where

$$d(x) := \prod_{j \geq 1} \cos(\pi x (-2)^{-j}) = \prod_{j \geq 1} \cos \frac{\pi x}{2^j}.$$

So if we define

$$\begin{aligned}
W_n &:= \Re(\omega^{(-2)^{-n}}M) \\
&= \begin{pmatrix} \frac{s}{2} \sin \frac{\pi x}{(-2)^n} - \frac{1}{2} \cos \frac{\pi x}{2^n} & \frac{b}{2} \cos \frac{\pi x}{2^n} - \frac{a}{2} \sin \frac{\pi x}{(-2)^n} \\ 0 & \cos(\pi x 2^{-n}) \end{pmatrix},
\end{aligned}$$

$$a := 1,$$

$$b := s,$$

then

$$\begin{pmatrix} 0 & b(x) \\ 0 & d(x) \end{pmatrix} = W_1 W_2 W_3 \dots$$

That is, our coveted Fourier coefficient $b(x)$ is just the upper right-hand element of the infinite matrix product of the W_n . This product converges for all s and x , and, miraculously, telescopes to a product of scalars. Telescopy in matrices means that you can find a nonzero matrix T_n so that

$$W_n T_{n+1} = T_n, \quad \forall \text{ relevant } n,$$

and thus,

$$W_1 W_2 \dots W_n T_{n+1} = T_1, \quad (\text{Telesc})$$

so as to provide useful relationships among the elements of the product.

With some luck and much computer algebra, we find a successful column vector

$$\begin{aligned} T_n &:= \begin{pmatrix} (-2)^{n-1} \frac{(3a - bs) \cos(\pi x (-2)^{1-n}) + (as - b) \sin(\pi x (-2)^{1-n})}{s^2 - 3} \\ (-2)^{n-1} \sin(\pi x (-2)^{1-n}) \end{pmatrix} \\ &= \begin{pmatrix} -(-2)^{n-1} \cos(\pi x (-2)^{1-n}) \\ (-2)^{n-1} \sin(\pi x (-2)^{1-n}) \end{pmatrix}, \end{aligned}$$

when a and b are specialized for $z(t)$. If we define b_n and d_n via

$$W_1 W_2 \dots W_n =: \begin{pmatrix} (-2)^{-n} \prod_{j=1}^n \left(\cos \frac{\pi x}{(-2)^j} - s \sin \frac{\pi x}{(-2)^j} \right) & b_n(x) \\ 0 & d_n(x) \end{pmatrix},$$

where

$$\begin{pmatrix} b(x) \\ d(x) \end{pmatrix} = \lim_{n \rightarrow \infty} \begin{pmatrix} b_n(x) \\ d_n(x) \end{pmatrix},$$

then equation (Telesc) becomes

$$\begin{pmatrix} (-2)^n b_n(x) \sin \frac{\pi x}{(-2)^n} - \cos \frac{\pi x}{2^n} \prod_{j=1}^n \left(\cos \frac{\pi x}{(-2)^j} - s \sin \frac{\pi x}{(-2)^j} \right) \\ (-2)^n \sin(\pi x (-2)^{-n}) d_n(x) \end{pmatrix} = \begin{pmatrix} -\cos \pi x \\ \sin \pi x \end{pmatrix}.$$

Letting $n \rightarrow \infty$, and solving for $b(x)$ and $d(x)$,

$$\begin{aligned} b(x) &= \frac{\prod_{j \geq 1} \left(\cos \frac{\pi x}{(-2)^j} - s \sin \frac{\pi x}{(-2)^j} \right)}{\pi x} - \frac{\cos \pi x}{\pi x} \\ &= \frac{\sin \pi x}{\pi^2 x^2} \prod_{j \geq 1} \left(1 - s \tan \frac{\pi x}{(-2)^j} \right) - \frac{\cos \pi x}{\pi x}, \\ d(x) &= \frac{\sin \pi x}{\pi x}, \end{aligned}$$

where we divided the infinite product by

$$\frac{\pi x}{\sin \pi x} \prod_{j \geq 1} \cos(\pi x (-2)^{-j}) = \frac{d(x)}{d(x)} = 1.$$

4 Proof of (1) and (2)

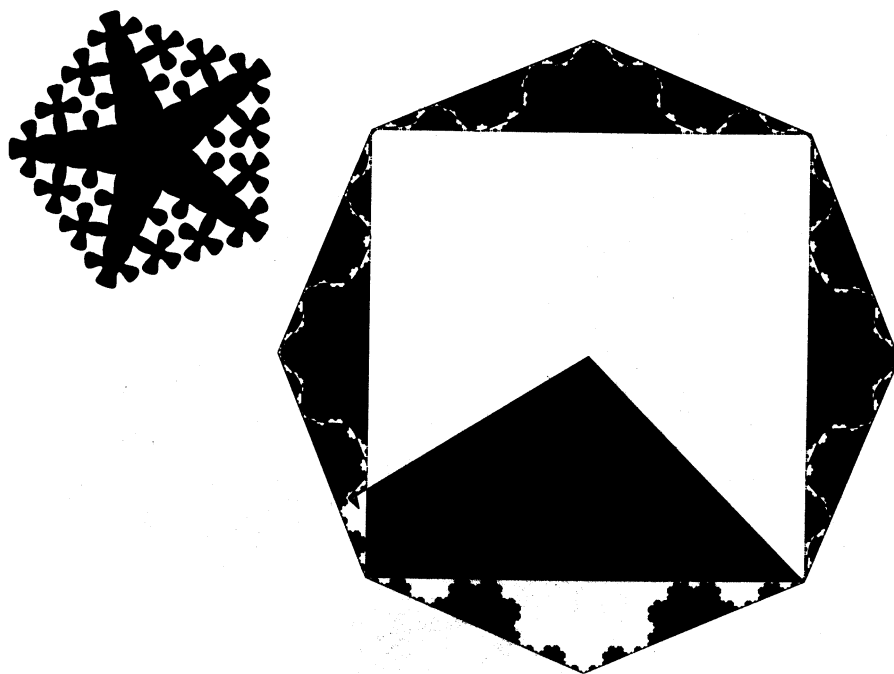
When $c = 1$, the $\cos(\pi x)/\pi x$ magically cancels from $A(x)$, and the Fourier identity is, at last,

$$z_m(t) = \frac{\sin \pi/m}{\pi^2} \sum_k (-)^k \frac{e^{i\pi(k+1/m)t}}{(k+1/m)^2} \prod_{n \geq 1} 1 - s \tan \frac{\pi(k+1/m)}{(-2)^n}.$$

(If $c \neq 1$, we still get (the numerator of) m congruent arcs, but displaced radially, and separated by step discontinuities, by virtue of adding the Fourier series of the “ m -gon” which is “swept” by lingering at each vertex for $1/m^{\text{th}}$ of the period, and then jumping to the next.)

So now we can write down mysterious-looking things like

$$\sum_k (-)^k \frac{e^{i\pi(k+1/m)/3}}{(k+1/m)^2} \prod_{n \geq 1} 1 - s \tan \frac{\pi(k+1/m)}{(-2)^n} = \frac{\pi^2 \left(\frac{s+i}{is+3} + \cot \frac{\pi}{m} \right)}{\sin \pi/m},$$



from the value of $z(1/3)$.

Figure 1, small: $z_5(-1,t)=z_{-5}(1,-t)$, a pentagon indented by Cesàro's spacefilled half-squares. Shown is the sum of Fourier terms (rotating vectors) $-32 \leq k < 31$, for $-5 \leq t \leq 5$, a full period. This is a low pass filtration of motion, rather than space, so the appearance of cusps merely indicates a momentarily vanishing time derivative of the partial sum.

Large: $z_4(\tan \pi/8, t) \sim z_8(-\tan \pi/8, 2t)$, *i.e.*, a bulged square = an indented octagon, if $|s|=\sqrt{2}-1$. Superposed is a Fourier approximation, partially swept ($-1 \leq t \leq 108/23$) by the sum of 18 rotating vectors, the first few of which are discernable at about "8 o'clock" ($t=108/23$). From the center, their indices run $0, -1, 1, -2, \dots, 8, -9$.



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Figure 2: Another way to approximate Cesàro's halfsquare filler. The values $-z_3(1, (7k-2557)/2560)$ for $k \in \{0, 1, \dots, 730\}$, along with the origin, connected in a self-contacting, but noncrossing polygon. The values were initially computed as exact complex rationals by a Lisp program analogous to the FOU function. But for graphics, floating point approximation proves much faster, providing $n/2$ bits of both x and y for n levels of recursion.

The graph of $z_{-m}(t)$ is that of $z_m(t)$ traced backward, and inside out, which can be seen by negating k in the bilateral series (figure 1, small):

$$z_m(s, t) = z_{-m}(-s, -t).$$

When $s = \tan(\pi/2m)$, bulging the sides of an m -gon is the same as indenting the sides of a (properly scaled and rotated) $2m$ -gon (figure 1, large):

$$z_m(s, t) = \frac{e^{-i\pi/2m}}{2} \sec \frac{\pi}{2m} z_{2m}(-s, 2t+1), \quad s = \tan \frac{\pi}{2m}.$$

For this s , the infinite product vanishes in the odd terms of the Fourier series for $z_m(s, t)$, and simple trigonometry shows that the even terms coincide with the right-hand side, above.

For rational t , we repeat the procedure we used for $z(0)$, recursing until the inevitable recurrence of some previous value (figure 2). Modern Lisps are ideally suited when s is a (perhaps complex) rational, since arbitrary precision complex rationals are a built-in data type. Likewise, some Lisps now have elaborate hash mechanisms to facilitate loop detection. Thus, a Lisp function, looking at $s = 1$, quickly ran off $z(t)$ for a few hundred rational t , revealing the interesting contrast between $z(1/130)$ and $z(2/261)$, hence identities (1) and (2). (The $(-)^k$ is omitted, since the odd terms vanish, since $s = 1 = \tan(\pi/4) = \tan(\pi/2m)$.)

The right-hand side of (2) can also be written

$$\frac{13\ 39841\ 33217702433766728023 + 2^7\ 3^2\ 5\ 11\ 71\ 28252454442382517i}{2^{86} + 4} \pi^2.$$

For general s , we will need computer algebra. A Macsyma program (next section) finds that identity (1) is the $s = 1$ case of

$$\begin{aligned} \sum_k (-)^k \frac{e^{i\pi(k+1/m)/130}}{(k+1/m)^2} \prod_{n \geq 1} 1 - s \tan \frac{\pi(k+1/m)}{(-2)^n} \\ = \frac{\pi^2}{\sin \pi/m} \left(\cot \frac{\pi}{m} + \frac{128s - i(s^2 - 1)(s^2 + 1)^3}{2(s^2 + 1)^3 + 128} \right). \end{aligned}$$

The constant term in the denominator is 130 because, for $s = 0$, the arc degenerates to the line segment $z(t) = it$, $-1 \leq t \leq 1$.

As you might expect, the generalization of (2) is somewhat worse, even with $m = 2$:

$$\begin{aligned} & \frac{4}{\pi^2} \sum_{k > -\infty} (-)^k \frac{e^{i\pi(2k+1)/261}}{(2k+1)^2} \prod_{n \geq 2} 1 + s \tan \frac{2k+1}{(-2)^n} \pi \\ & (s-i)^5 (s+i)^3 \left(s^{77} + 2is^{76} + 34s^{75} + 76is^{74} + \dots \right. \\ & \quad \left. - 1123939790088261808331s - 2i \frac{2^{84} - 1}{261} \right) + 2^{84}s \\ & = \frac{\quad}{(s-i)^{41}(s+i)^{43} + 2^{84}}. \end{aligned}$$

We can get an alternative Fourier expansion for $z(t)$ by reverting to the original product form of $b(k+1/m)$ in

$$\begin{aligned} z(t) &= z_m(t) - \cot \frac{\pi}{m} \\ &= \sum_k \frac{e^{i\pi(k+1/m)t}}{\pi(k+1/m)} \prod_{n \geq 1} \left(\cos \frac{\pi(k+1/m)}{(-2)^n} - s \sin \frac{\pi(k+1/m)}{(-2)^n} \right) - \cot \frac{\pi}{m}, \end{aligned}$$

and then letting $m \rightarrow \infty$. The $k = 0$ term offsets the $\cot \pi/m$, leaving

$$\begin{aligned} z(t) &= it + \frac{s}{3} + \sum_{k \neq 0} \frac{e^{i\pi kt}}{\pi k} \prod_{n \geq 1} \left(\cos \frac{\pi k}{(-2)^n} - s \sin \frac{\pi k}{(-2)^n} \right), \quad -1 \leq t \leq 1, \\ &= \frac{s}{3} + \sum_{k \neq 0} \frac{e^{i\pi kt}}{\pi k} \left((-1)^{k-1} + \prod_{n \geq 1} \cos \frac{\pi k}{(-2)^n} - s \sin \frac{\pi k}{(-2)^n} \right), \quad |t| < 1, \end{aligned}$$

using the “sawtooth” Fourier series

$$t = i \sum_{k \neq 0} (-)^k \frac{e^{i\pi kt}}{\pi k} = -2 \sum_{k \geq 1} \frac{\sin(\pi k(t+1))}{\pi k}, \quad -1 < t < 1.$$

This $z(t)$ coincides with $z_2(t)$ for $-1 < t < 1$, but then discontinuously jumps from i back to $-i$ instead of repeating the arc rotated end for end. Thus the centroid of the arc is the constant term, $s/3$.

Because Fourier approximations interpolate the midpoint of jump discontinuities, we can say that

$$\sum_{k \neq 0} \frac{(-)^k}{k} \prod_{n \geq 1} \left(\cos \frac{\pi k}{(-2)^n} - s \sin \frac{\pi k}{(-2)^n} \right) = -\frac{\pi s}{3}.$$

We can also say weird things like

$$\sum_{k \neq 0} \frac{e^{i\pi k/3}}{k} \prod_{n \geq 1} \left(\cos \frac{\pi k}{(-2)^n} - s \sin \frac{\pi k}{(-2)^n} \right) = \pi(s+i) \left(\frac{1}{is+3} - \frac{1}{3} \right),$$

again using our value of $z(1/3)$.

5 The Macsyma Function

It would have been clearer to name it $Z(T)$, but, as we well know, all programs are named **F00**, in honor of Fourier.

```
(C50) dispfun(fou,bar)$
```

```
(E50) FOU(T, [V]) := (V : IF V = [] THEN [1, 0] ELSE V ,
```

```
GFACOR(SUBST(V, 'V, BAR )))
```

```
(E51) BAR_T := (BAR_T :  $\frac{V_2 - '(V)_2}{'(V)_1 - V_1}$ ,
```

```
BAR_T :
```

```
(i S - 1) FOU(- 2 T - 1, (V .  $\begin{bmatrix} \frac{i S - 1}{2} & \frac{S - i}{2} \\ 0 & 1 \end{bmatrix}$ )) )
```

```
IF T < 0 THEN -----
```

```
2
```

$$+ \frac{S - i}{2}$$

$$\begin{aligned}
 & (-iS - 1) \text{FOU}(1 - 2T, (V \cdot \begin{bmatrix} -iS - 1 & S + i \\ 2 & 2 \\ 0 & 1 \end{bmatrix})) \\
 & \text{ELSE } \frac{\frac{1}{2} \left(\frac{S + i}{2} \right)}{2}
 \end{aligned}$$

(To convince (normally case-insensitive) Macsyma to say `i` instead of `%I`, `alias(\i,%i)`. You still have to input `\i` or `%i`, but the output is prettier.)

The optional argument `v` maintains the top row of the cumulative product of the matrices `M`. `BAR` is the memoizing hash array. Normally, such array elements redefine themselves automatically on the first call, but in this case, they need to redefine themselves *twice*: first to set the trap for looping back to the same `t`, and again to record the answer when either the trap springs or some subanswer was found prerecorded. Springing the trap is just evaluating the difference quotient in the temporary definition of `BAR`. This effectively simulates infinite recursion by computing the fixed point of the pending matrix transformations. In older Macsymas, the simplifier (but not the evaluator) neglects to perform the part selection when given a subscripted list, for example, `'[a,b][1]`. So instead of `subst(v,'v,bar[t])`, one might write `eval(bar[t])` in the definition of `FOU`. However, as Lisp teaches us, indiscriminate evaluation is a poor idea (what if `s` happened to be bound?), so a cleaner program might `subst(map("=", '[v[1],v[2]],v),bar[t])` instead.

Testing it:

```
(C52) fou(1);
Time= 122 msec.
(D52)
```

`i`

```
(C53) fou(0);
Time= 61 msec.
(D53)
```

`s`

```
(C54) fou(1/130);
Time= 10108 msec.
```

$$\text{(D54)} \quad - \frac{i(S^8 + 2S^6 - 2S^2 + 128iS - 1)}{2(S^2 + 5)(S^4 - 2S^2 + 13)}$$

Of course, subsequent evaluations of `fou(1/130)` would be quicker than the exhibited (first) evaluation of `fou(0)`.

Programming quiz: What would happen if you aborted an evaluation with several `BAR` traps unsprung? Answer: Something bad, the next time you tried to evaluate one of those booby-trapped entries. If you retried the same top-level call to `FOU`, you would divide 0 by 0. Much worse, if you reach a half-computed `BAR` entry with a different `v` accumulated in `FOU`, you will simply get a wrong answer, and poison the `BAR` cache.

Like modern Lisps, Macsyma provides the indispensable `UNWIND_PROTECT` form with which a more carefully written `BAR` can detect an abnormal exit and unset the unsprung trap.

6 Adumbration of (3)

Instead of a dyadic fractal arc from $-i$ via s to i , let $Z(t)$ be the triadic from $-i$ via $w - i$ and $i - w$ to i , for some complex w . For convergence, all three segments must have length < 2 , that is, $|w - i| < 1$. For $w = i \pm 1/\sqrt{3}$, $Z_3(t)$ is the triadic dragon that sweeps the “fudgeflake” [Man, p. 73]. [Man, p. 67] incorrectly claims (modulo typos) that if $w = (i \pm 1)/2$, $Z_2(t)$ runs back and forth along the “twindragon river”, which is the skin of one side of the traditional (dyadic) dragon curve. In fact, this skin requires a tetradic “isosceles” construction, rather than a “scalene” triadic one.

If w is pure imaginary, the arc is simply linear, but swept in different fashion depending on $|w|$. For $w = 2i/3$, the sweep is linear and $Z(t) = it$. But if $w = i$, the central segment vanishes, and $Z(t)/i$ (plotted versus t) becomes the devil’s staircase! [Man, p. 82]. Thus, $Z_m(t)$ would *continuously* trace out an ordinary regular m -gon, but with 0 instantaneous velocity “almost always”.

The recursive definition is

$$Z(t) := \begin{cases} -\frac{iw}{2}Z(3t+2) + \frac{w}{2} - i, & -1 \leq t \leq -1/3; \\ (iw+1)Z(3t), & -1/3 \leq t \leq 1/3; \\ -\frac{iw}{2}Z(3t-2) - \frac{w}{2} + i, & 1/3 \leq t \leq 1. \end{cases}$$

The coefficients of $Z()$ for each case determine the three corresponding matrices M_- , M_0 , M_+ , which map $[-i, i]$ onto the three segments. Then, analogous to the dyadic case,

$$2 \prod_{j \geq 1} \frac{\omega^{-3^{-j}} M_- + M_0 + \omega^{3^{-j}} M_+}{3}$$

provides the Fourier coefficient $A(x)$ of $e^{i\pi x t}$, when $\omega := e^{-i\pi x}$. (Since the arc is symmetric under 180° rotation, we need not worry about traversing subarcs backward, hence the lack of $(-3)^{-j}$.)

Another telescoping miracle reduces the matrix product to a scalar one, and

$$Z_m(t) = \frac{1}{\pi} \sum_{k > -\infty} \frac{\prod_{n \geq 1} 1 + 2iw \left(\sin \frac{k + 1/m}{3^n} \pi \right)^2}{k + 1/m} e^{i\pi(k+1/m)t}$$

repeats the fractal around an m -gon. Since, for $-1 \leq t \leq 1$, $Z_m(t) = \cot(\pi/m) + Z(t)$,

$$\sum_{k > -\infty} \frac{1 - e^{i\pi(k+\phi)t} \prod_{n \geq 1} 1 + s \left(\sin \frac{k+\phi}{3^n} \pi \right)^2}{k + \phi}, \quad -1 \leq t \leq 1,$$

is independent of ϕ , and an odd function of t , given the arc's aforementioned symmetry.

Identity (3) simply states that the even part is 0. (For $t \neq 0$, the series is conditionally convergent, and needs to be summed sufficiently symmetrically that $\sum_k 1/(k + \phi) = \pi \cot \pi \phi$.)

More generally, $Z(\text{rational } t)$ will be a rational function of w whose degree depends on the balanced ternary expansion of t . For example,

$$\begin{aligned} Z(-1) &= -i, & Z(0) &= 0, & Z(1) &= i, \\ Z\left(\frac{1}{3}\right) &= i - w, & Z\left(\frac{1}{2}\right) &= -i - \frac{4}{w + 2i}, & Z\left(\frac{2}{3}\right) &= i - \frac{w}{2}, \\ Z\left(\frac{1}{5}\right) &= -i - \frac{2}{w + i}, & Z\left(\frac{2}{5}\right) &= i + \frac{4w}{w^2 - 4}, & Z\left(\frac{3}{5}\right) &= -\frac{1}{w + i}, \\ Z\left(\frac{1}{7}\right) &= i + \frac{4w}{w^2 + iw - 2}, & Z\left(\frac{3}{7}\right) &= \frac{w - 2i}{w^2 + iw - 2}, \\ Z\left(\frac{1}{9}\right) &= i - 2w - iw^2, & Z\left(\frac{5}{9}\right) &= i - w - \frac{i}{2}w^2, \\ Z\left(\frac{3}{11}\right) &= -\frac{(w - i)(w^2 - 2iw + 4)}{w^4 - w^2 - 2iw + 4}, & Z\left(\frac{3}{13}\right) &= \frac{w - i}{w^2 - 1}, \\ Z\left(\frac{1}{15}\right) &= \frac{(w - i)^2}{w + i}, & Z\left(\frac{2}{15}\right) &= -\frac{(w - i)(w - 2i)^2}{w^2 - 4}, \end{aligned}$$

$$Z\left(\frac{1}{18}\right) = i \frac{(w-2i)(w-i)^2}{w+2i}, \quad Z\left(\frac{1}{27}\right) = (w-i)^3,$$

$$Z\left(\frac{27}{40}\right) = -\frac{w-2i}{w(w-i)^3+2}, \quad Z\left(\frac{27}{41}\right) = \frac{1}{w^3-iw^2-w-i},$$

etc. ($Z(3/11)$ and $Z(1/15)$ were originally misprinted.)

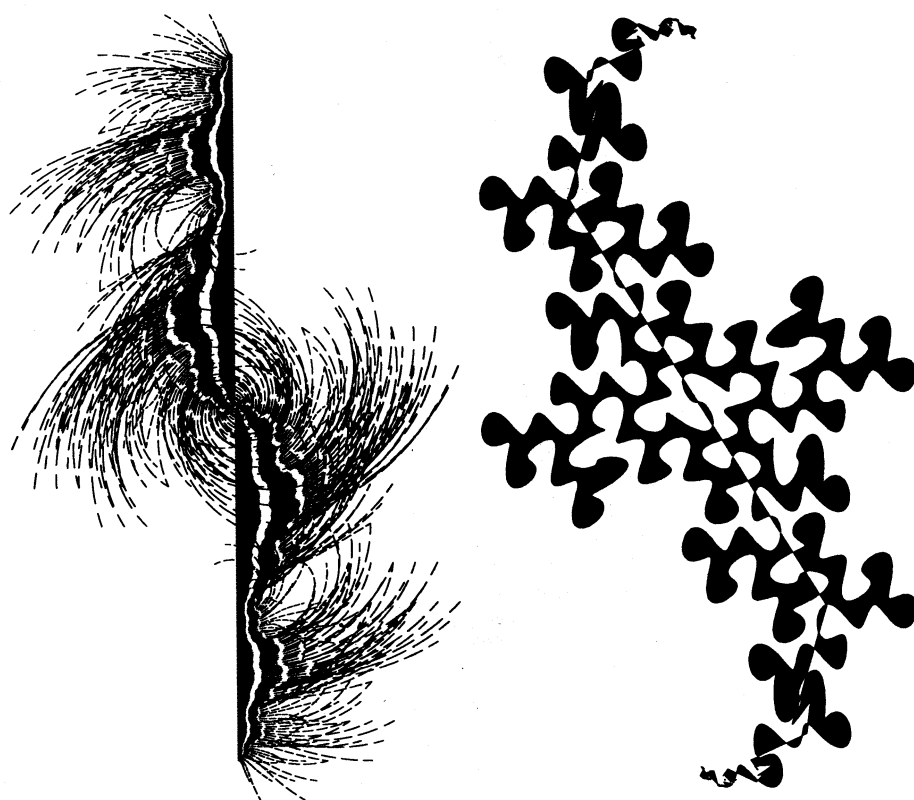


Figure 3, frizzy: The trajectories of $Z(k/243)$, $k \in \{-243, -242, \dots, 243\}$, as w sweeps from $2i/3$ (line segment $=Z(t)$) to $1/\sqrt{3}+i$ (spacefilling triadic dragon $=Z(t)$), along the arc $|w|=2|w-i|$, that is, $|Z(-1)-Z(-1/3)|=|Z(-1/3)-Z(1/3)|$, the “isosceles” Koch constructions. The gaps show $Z(t)$ at intermediate dimensions. Despite the requirement that the points flow from the segment to completely cover an area, there are many confluences—spacefilling functions visit infinitely many points three times.

Knobby: The filled (triadic dragon) area, this time approximated by Fourier terms $0, -1, 1, -2, \dots, 63, -64$ of $Z_2(t)$, $-1 \leq t \leq 1$. During this half-period, the fundamental swept from “5 o'clock” to “11 o'clock”, and the tip of the -127^{th} harmonic swept the meandering curve from “6 o'clock” to “12 o'clock”. $Z_2(0)=0$ via term k cancelling term $-k-1$, but $t=\pm 1$ introduces an alternating sign, effectively doubling each term. These are visible as the rotationally symmetric ($t=-1$ and $t=1$) broken lines from the center of the diagram. Each segment is hinged in the middle, and folds into two counter-rotating vectors as t departs either endpoint.

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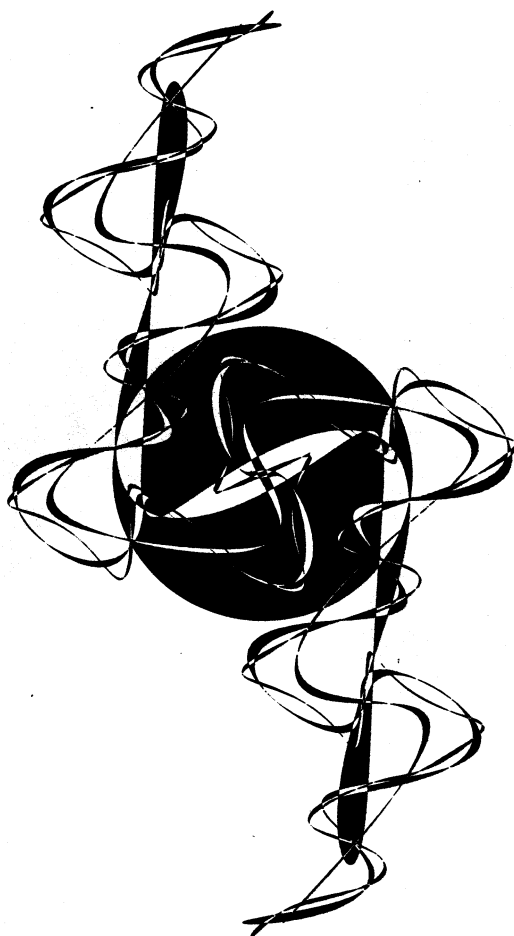


Figure 4: The regions swept by harmonics 1, 5, 13, 29, and 61 during a full period of $Z_2(t)$. Observe that they have one lobe (circle), 5 lobes, 13 lobes, ..., respectively.

It would be interesting to know what subspace of the rational functions this represents, but since $Z(t)$ is continuous, we know that for arbitrarily close t , these functions must arbitrarily well approximate each other in the disk $|w - i| < 1$. For example, $27/41 < 2/3 < 27/40$, and

$$\begin{aligned} Z\left(\frac{27}{40}\right) - Z\left(\frac{2}{3}\right) &= \frac{w(w-2i)(w-i)^3}{2(w(w-i)^3 + 2)} \\ &= \frac{(w-i)^3}{4} + \frac{(w-i)^5}{4} - i\frac{(w-i)^6}{8} - \frac{(w-i)^7}{8} + \dots, \\ Z\left(\frac{2}{3}\right) - Z\left(\frac{27}{41}\right) &= -\frac{w}{2} \frac{(w-i)^3}{w^3 - iw^2 - w - i} \\ &= \frac{(w-i)^3}{4} + \frac{(w-i)^5}{4} + i\frac{(w-i)^6}{8} + \frac{(w-i)^7}{8} + \dots. \end{aligned}$$

We know that, for $|t| \leq 1/3$, there will be a factor of $w - i$, because of the devil's staircase property; and, of course, $Z(t) - it$ will contain a factor of $3w - 2i$. Also, the degree in w (expansion about ∞) appears to be $-\alpha_3$ in the rational factorization $|t| = 2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} \dots$.

As w describes a continuous arc from $i2/3$ to $i+1/\sqrt{3}$, the arc $Z(t)$ continuously deforms from a one-dimensional line segment to a two-dimensional "triadic dragon". The functions of w above are the (smooth) trajectories of specific fixed t during the dimension boost. Only a Cantor set of the t wind up on the boundary of the dragon.

This entire discussion of rational functions from rational t can be repeated for the dyadic $z(t)$ underlying identities (1) and (2).

We can also repeat the trick of taking $m \rightarrow \infty$ in $Z(t) = Z_m(t) - \cot(\pi/m)$ to get a nicer but discontinuous

$$Z(t) = 2i \sum_{k \geq 1} \frac{\sin \pi k t}{\pi k} \left((-1)^{k-1} + \prod_{n \geq 1} 1 + 2iw \sin^2 \left(\frac{\pi k}{3^n} \right) \right), \quad -1 < t < 1.$$

7 Nonlocal Derangement

While investigating a so-called q -analogue of $\sin$, I conjectured the generalized addition formula,

$$\sin \sum_{k=1}^n a_k = \sum_{k=1}^n \sin a_k \prod_{\substack{j=1 \\ j \neq k}}^n (\cos a_j + \sin a_j \cot(b_j - b_k)), \quad (n \text{ angles})$$

for arbitrary n , a_k and b_k , provided the b_k are distinct mod π . (Note the similarity of the product to the first formula for $b(x)$.) The special case $a_k := \pi/2$ appears in Whittaker and Watson [5]. More generally, with $a_k := \arctan y/x$,

$$\sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n (x + y \cot(b_j - b_k)) = \frac{(x^2 + y^2)^{n/2}}{y} \sin \left(n \arctan \frac{y}{x} \right);$$

or, if $a_k := z + \pi k/n$,

$$\sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \left(\cot \left(z + \frac{\pi j}{n} \right) + \cot(b_j - b_k) \right) = -2^{n-1} \frac{\cos \left(n \left(z + \frac{\pi}{2} \right) \right)}{\sin(nz)}, \quad n \geq 1.$$

In $(n \text{ angles})$, replacing a_k and b_k by ϵa_k and ϵb_k , expanding at $\epsilon = 0$, and equating first order terms:

$$\sum_{k=1}^n a_k = \sum_{k=1}^n a_k \prod_{\substack{j=1 \\ j \neq k}}^n 1 + \frac{a_j}{b_j - b_k}. \quad (n \text{ terms})$$

As we shall see, this is actually *equivalent* to $(n \text{ angles})$, so in proving the latter, Vilmos Totik proved both [Ism].

Totik observes that the coefficient of $(b_1 - b_2)^{-1}$ is

$$a_1 a_2 \left(\prod_{j=3}^n \left(1 + \frac{a_j}{b_j - b_2} \right) - \prod_{j=3}^n \left(1 + \frac{a_j}{b_j - b_1} \right) \right),$$

which is $O(b_1 - b_2)$. Therefore, the right side of $(n \text{ terms})$, as a function of b_1 , has no poles, and is therefore constant; likewise for the other b_k . To

determine that constant, choose the b_k arbitrarily large, reducing (n terms) to a tautology.

To recover (n angles), range the sums and products in (n terms) over $[0, n]$ instead of $[1, n]$, then subtract the original (n terms), leaving

$$a_0 = a_0 \prod_{j=1}^n \left(1 + \frac{a_j}{b_j - b_0}\right) + \sum_{k=1}^n a_k \frac{a_0}{b_0 - b_k} \prod_{\substack{j=1 \\ j \neq k}}^n 1 + \frac{a_j}{b_j - b_k}.$$

Then change variables: $a_0 := 1, a_k := \alpha_k \beta_k, b_0 := 0, b_k := -\beta_k$:

$$1 - \prod_{j=1}^n 1 - \alpha_j = \sum_{k=1}^n \alpha_k \prod_{\substack{j=1 \\ j \neq k}}^n 1 - \frac{\alpha_j \beta_j}{\beta_j - \beta_k}. \quad (n \text{ factors})$$

Then change variables again: $\alpha_k := 1 - e^{-2ia_k}, \beta_k := e^{ib_k}$, multiply through by

$$\frac{\prod_{j=1}^n e^{ia_j}}{2i},$$

combine the products in the summand, using

$$\frac{\beta_j}{\beta_j - \beta_k} = \frac{1}{2} + \frac{1}{2} \frac{\beta_j + \beta_k}{\beta_j - \beta_k},$$

and you are done.

But there ain't nothin' new under the sun. [Whi] gives (as an exercise!):

$$\cos \sum_{k=1}^n (a_k - b_k) = \prod_{j=1}^n \frac{\sin(x - b_j)}{\sin(x - a_j)} - \sum_{k=1}^n \cot(x - a_k) \sin(a_k - b_k) \prod_{\substack{j=1 \\ j \neq k}}^n \frac{\sin(a_k - b_j)}{\sin(a_k - a_j)}. \quad (\text{W\&W})$$

For small a_k, b_k , the expansion to first order is equivalent to (n factors), under a rational function change of variables. This transformation is invertible, making it likely that (W&W) is merely equivalent to all the rest.

8 Proof of (4) and (5)

Identity (4) is just (n factors), with $n \rightarrow \infty$, $\alpha_k := ap^k$, and $\beta_k := c + q^k$.

Identity (5) also comes from (n factors), and is actually a bilateral series folded in half by coalescing term n with term $-n-1$. It is the special case $\phi = 1/2, b = i$ of

$$\sum_{n>-\infty} \frac{(-)^n \cos \pi \sqrt{(n + \phi + \frac{a}{b})(n + \phi + ab)}}{(n + \phi)(n + \phi - a)}$$

$$= \frac{\pi}{a} \frac{\frac{\sin(\pi a) \cos(\pi(\phi - a))}{\sin(\pi \phi)} - 2 \sin^2 \frac{\pi a(b+1)}{2\sqrt{b}}}{\sin(\pi(\phi - a))}.$$

This in turn caps a longish sequence of specializations and renamings of coefficients, with both α_k and β_k initially general reciprocal quadratics, that is, of the form $c_1/(k+c_2)(k+c_3)$. These create rational functions in the bilateral infinite product, which simplify to the cosine via the factorial (Γ) reflection formula.

Identity (5) was sufficiently peculiar to merit a Taylor expansion for small a . Indeed, both sides begin

$$-\frac{\pi^5}{12}a^2 - \frac{\pi^7}{40}a^4 - \frac{\pi^9}{96}a^6 - \frac{17\pi^{11}}{4032}a^8 - \frac{409267\pi^{13}}{239500800}a^{10} - \dots$$

For some analogous formulas from (n terms), and generalizations of the cosines to Bessel functions, see Gosper [1]. George Gasper was first to recognize that identities like (5) are discrete analogs of Sonine's Bessel integrals [Wat].

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