$$\int_{n/4}^{m/6} \ln\Gamma(z)dz$$

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Abstract: We evaluate $\int \ln\Gamma(z)dz$, which could be notated $\psi_{-2}(z)$, between rational limits with denominators $1, 2, 3, 4$, and 6. The simplest interesting result is (using $t!$ for $\Gamma(t+1)$)

$$\int_0^{1/2} \ln t! dt = \frac{\gamma + 3\ln\pi}{8} - \frac{\ln 2}{6} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2} \approx -.04285374065.$$

We note that all integrals of the form $\int z^n \ln z! dz$ are expressible in terms of $\zeta'(-n, z)$, the first derivative of the Hurwitz zeta function.

Introduction

We establish

$$\int_0^{1/2} \ln t! dt = \frac{\gamma + 3\ln\pi}{8} - \frac{\ln 2}{6} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2} \approx -.04285374065,$$

$$\int_0^{1/3} \ln t! dt = \frac{2\gamma + 5\ln 2\pi}{18} - \frac{25\ln 3}{72} - \frac{\sqrt{3}\pi}{54} + \frac{\sqrt{3}\psi_1(1/3)}{36\pi} - \frac{2\zeta'(2)}{3\pi^2} - \frac{1}{3} \approx -.02296612858214,$$

$$\int_0^{1/4} \ln t! dt = \frac{3\gamma + 7\ln\pi - 9\ln 2}{32} - \frac{9\zeta'(2)}{16\pi^2} + \frac{\lambda}{4\pi} - \frac{1}{4} \approx -.014099944308,$$

$$\int_0^{1/6} \ln t! dt = \frac{10\gamma - 23\ln 3 + 22\ln\pi}{144} - \frac{\sqrt{3}\pi}{36} + \frac{\sqrt{3}\psi_1(1/3)}{24\pi} - \frac{5\zeta'(2)}{12\pi^2} - \frac{1}{6} \approx -.006818642158464,$$

where

$$\lambda := \text{Catalan's constant} = \frac{1}{8}\left(\psi_1\left(\frac{1}{4}\right) - \pi^2\right) = \frac{1}{16}\left(\psi_1\left(\frac{1}{4}\right) - \psi_1\left(\frac{3}{4}\right)\right) \approx .9159655941772,$$

which, with the Γ multiplication formula and its limit, the Barnes-Alexeiewsky recurrence,

$$\int_{z-1}^{z} \ln t! dt = \ln\sqrt{2\pi} - (1 - \ln z)z,$$

suffice for $\int_{n/4}^{m/6}$. *E.g.*,

$$\int_0^{2/3} \ln t! dt = \frac{\gamma + 10\ln 2 + 4\ln\pi}{9} - \frac{49\ln 3}{72} + \frac{\sqrt{3}\pi}{54} - \frac{\sqrt{3}\psi_1(1/3)}{36\pi} - \frac{2\zeta'(2)}{3\pi^2} - \frac{2}{3} \approx -.06178124243647$$

$$\int_0^{3/4} \ln t! dt = \frac{3\gamma + 15\ln\pi - 33\ln 2}{32} + \frac{3\ln 3}{4} - \frac{9\zeta'(2)}{16\pi^2} - \frac{\lambda}{4\pi} - \frac{3}{4} \approx -.06959909378,$$

$$\int_0^{5/6} \ln t! dt = \frac{10\gamma - 119\ln 3 + 70\ln\pi}{144} + \frac{5\ln 5 - 2\ln 2}{6} + \frac{\sqrt{3}\pi}{36} - \frac{\sqrt{3}\psi_1(1/3)}{24\pi} - \frac{5\zeta'(2)}{12\pi^2} - \frac{5}{6}$$
$$\approx -.07570064608869,$$

The key lemma is an expansion of the general polylogarithm Li_k near 1.

Lemmas:

[Cohen, 1992] gives, without proof,

$$\operatorname{Li}_k(e^{-z}) = \begin{cases} (-k)!z^{k-1} + \sum_{n\ge 0} \frac{\zeta(k-n)}{n!}(-z)^n, & \text{if } k \ne \lfloor k \rfloor \text{ or } k \le 0; \text{ else} \\ (H_{k-1} - \ln z)\,\frac{(-z)^{k-1}}{(k-1)!} + \sum_{n\ge 0, \ne k-1} \frac{\zeta(k-n)}{n!}(-z)^n, & H_{k-1} := \Psi_0(k) + \gamma; \end{cases} \tag{Li1}$$

where

$$\operatorname{Li}_k(z) := \sum_{n\ge 1} \frac{z^n}{n^k}.$$

([Gosper, 1990] gave (Li1) for negative integer k.) Differentiating ∂k at 1,

$$\begin{aligned} \sum_{n\ge 1} \frac{(\ln n)e^{-nz}}{n} &= -\left.\frac{\partial \operatorname{Li}_k(e^{-z})}{\partial k}\right|_{k=1} \\ &= -\sum_{n\ge 1} \frac{\zeta'(1-n)}{n!}(-z)^n + \frac{(\gamma + \ln z)^2}{2} + \gamma_1 + \frac{\pi^2}{12}, \end{aligned} \tag{Li1$'$}$$

where

$$\begin{aligned} \gamma_1 &= \frac{(\ln 2\pi)^2 - \gamma^2}{2} + \zeta''(0) + \frac{\pi^2}{24} \\ &= \sum_{n\ge 1} \frac{\ln n}{n} - \frac{\left(\ln(n+1)\right)^2 - (\ln n)^2}{2} \approx -.07281584548367672, \end{aligned}$$

the first Stieltjes constant [Liang, 1972]. Differentiating the ζ reflection formula at a negative, even integer:

$$\zeta'(-2n) = (-)^n \frac{(2n)!\zeta(2n+1)}{2^{2n+1}\pi^{2n}}.$$

We can apply this to formula (Li1$'$) if we first eliminate $\zeta'(1-2n)$ by setting $z = 2\pi i t$ and taking the imaginary part. The resulting infinite series coincides with the Taylor expansion of $\ln\Gamma(t)$ at 0:

$$\begin{aligned} \ln\Gamma(t) &= \frac{1}{2}\left(\ln\Gamma(t) - \ln(-\Gamma(-t)) + \ln\frac{\pi}{t\sin\pi t}\right) \\ &= \frac{1}{2}\ln\frac{\pi}{t\sin\pi t} - \gamma t - \sum_{n\ge 1} \frac{\zeta(2n+1)}{2n+1} t^{2n+1} \\ &= \sum_{n\ge 1} \frac{\ln n}{\pi n}\sin 2\pi n t + \frac{1}{2}\left(\ln\frac{2\pi^2}{\sin\pi t} + \gamma\right) - (\gamma + \ln 2\pi)t, \qquad 0 < t < 1, \end{aligned}$$

which is (the hard part of) the Fourier expansion of $\ln\Gamma(t)$. Finishing the expansion, via

$$\ln\sin\pi t = -\sum_{n\geq 1}\frac{\cos 2\pi nt}{n} - \ln 2, \qquad 0<t<1, \tag{ln sin}$$

we get

$$\ln\Gamma(t) = \frac{\ln 2\pi}{2} + \sum_{n\geq 1}\frac{2(\gamma+\ln 2\pi n)\sin(2\pi nt)+\pi\cos(2\pi nt)}{2\pi n}, \qquad 0<t<1. \tag{Fou}$$

Differentiating (Li1) ∂k at 2 produces the same series as integrating equation (Li1′) dz:

$$\begin{aligned}\sum_{n\geq 1}\frac{(\ln n)e^{-nz}}{n^2} &= -\left.\frac{\partial\operatorname{Li}_k(e^{-z})}{\partial k}\right|_{k=2}\\ &= -\sum_{n\geq 2}\frac{\zeta'(2-n)}{n!}(-z)^n - \left(\frac{(\gamma+\ln z)(\gamma-2+\ln z)}{2}+\gamma_1+\frac{\pi^2}{12}+1\right)z - \zeta'(2),\end{aligned}$$

Thus

$$\sum_{n\geq 1}\frac{\pi\sin(2\pi nt)-2(\ln n)\cos(2\pi nt)}{4\pi^2 n^2} = \ln f_1(t-1) + (\gamma-1+\ln(2\pi))\binom{t}{2} + \frac{\zeta'(2)}{2\pi^2}, \tag{ψ_{-2}}$$

where

$$f_1(n) = 1^1 2^2 \ldots n^n, \qquad f_1(z) := (2\pi)^{-z/2}\exp\left(\binom{z+1}{2} + \int_0^z \ln t!\,dt\right),$$

which could equally well come from integrating (Fou) dt.

This series comes out in terms of $\zeta(2)$ and $\zeta'(2)$ for $t = j/4$ and $k/6$, yielding the identities asserted in the Introduction.

[Glaisher, 1895] denotes $\int \ln\Gamma(t)dt$ by ilg, but, if we can decide on the right constant of integration, it would seem logical to simply call it $\psi_{-2}(t)$, the first "negapolygamma". By parts, it is equivalent to $\int t\psi_0(t)dt$. f_1 is sometimes called Barnes's G function. $\zeta'(2)$ or simple functions thereof are sometimes called Glaisher's or Porter's constant [Knuth, 1976], [Pitteway, 1988].

It seems strange that, at rationals like $1/3, 1/4$, and $1/6$, the function which interpolates $1^1 2^2 \ldots n^n$, *i.e.*, $\int\ln\Gamma$ comes out in simpler constants, while the function which interpolates $1\cdot 2\cdots n$, *i.e.*, Γ does not. However, if we denote $\ln\Gamma$ by ψ_{-1}, this extends for subscripts -1 and -2 a pattern where $\psi_n(j/4$ or $k/6)$ simplifies iff n is even. What seems stranger is that the Fourier coefficients of $\ln\Gamma(t)$ should be elementary, while those of $\ln t$ are not.

Further results

(ψ_{-2}) can be repeatedly integrated by parts to give $\int_0^u t^n \ln\Gamma(t)dt$, but for $n>0$, the only u for which the series has an obvious closed form are integers and half integers. *E.g.*,

$$\int_0^{1/2} t\ln t! = \frac{\gamma}{48} + \frac{\ln\pi}{12} - \frac{4\zeta'(2)+7\zeta(3)}{32\pi^2} - \frac{1+\ln 2}{16} \approx -.01317042784872,$$
$$\int_0^{1/2} t^2\ln t! = \frac{\gamma+5\ln\pi}{192} - \frac{2\zeta'(2)+3\zeta(3)}{32\pi^2} + \frac{15\zeta'(4)}{32\pi^4} - \frac{37\ln 2}{1440} - \frac{1}{72} \approx -.00469460599089.$$

The earlier ψ_{-2} integrals, combined with the Γ and ψ reflection formulæ, yield $\int_{n/4}^{m/6} \ln\sin\pi t dt$. A superior alternative is to integrate equation (ln sin),

$$\int_0^u \ln\sin\pi t dt = -u\ln 2 - \frac{1}{2\pi}\sum_{n\geq 1}\frac{\sin 2n\pi u}{n^2}, \qquad 0\leq u\leq 1,$$

which gives trigammas for *all* rational u. *E.g.*,

$$\int_0^{1/5} \ln\sin\pi t dt = \frac{1}{25\pi}\left(\frac{4}{\sqrt{5}}\phi\pi^2 - \phi\psi_1\left(\frac{1}{5}\right) - \psi_1\left(\frac{2}{5}\right)\right)\sin\frac{\pi}{5} - \frac{\ln 2}{5} \approx -.2973633652639,$$

where

$$\phi := \frac{1+\sqrt{5}}{2} \approx 1.6180339887499.$$

This, with the Γ reflection formula, gives $\ln\Gamma$ integrals over intervals symmetric about integers or half integers:

$$\int_{1-z}^{z} \ln t! dt = z\ln z - (1-z)\ln(1-z) + (1-2z)(1-\ln\sqrt{\pi}) - \int_{1-z}^{1/2}\ln\sin\pi t dt.$$

E.g.,

$$\int_{2/5}^{3/5} \ln t! dt = \frac{\ln\pi - 3\ln 2}{10} + \frac{1}{25\pi}\left(\phi\psi_1\left(\frac{2}{5}\right) - \psi_1\left(\frac{1}{5}\right) + \frac{2\pi^2}{\phi\sqrt{5}}\right)\sin\frac{\pi}{5} + \frac{3\ln 3 - \ln 5 - 1}{5} \approx -.0238446115029.$$

With repeated integration of (ln sin) by parts, we get $\int_0^u t^n \ln\sin\pi t dt$ in terms of polygammas for all positive integer n and rational u. E.g.,

$$\int_0^{1/8} t\ln\sin\pi t dt = \frac{\sqrt{2}}{2048\pi^2}\left(\psi_2\left(\frac{1}{8}\right) - 4\psi_1\left(\frac{1}{8}\right)\right) + \frac{5+4\sqrt{2}}{512}\pi + \frac{2\sqrt{2}-1}{64\pi}\lambda + 7\frac{37+16\sqrt{2}}{1024\pi^2}\zeta(3) - \frac{\ln 2}{128}$$
$$\approx -.01130943023.$$

The ψ_{-3} integral (equivalent to $\int t^2\psi_0(t)dt$) over the general unit interval appears to require the ψ_{-2} integral over the general interval:

$$\int_{z-1}^{z} t\ln t!dt = \int_0^z \ln t!dt + \frac{z-2}{2}z\ln z - \frac{z^2}{4} + z - \frac{\ln 2\pi}{3} + \frac{\zeta'(2)}{2\pi^2}.$$

E.g., employing the $\psi_{-2}(1/6)$ result from the Introduction,

$$\int_{-5/6}^{1/6} t\ln t!dt = \frac{\sqrt{3}}{24\pi}\psi_1\left(\frac{1}{3}\right) - \frac{\gamma + 13\ln 2\pi}{72} - \frac{1+\ln 3}{144} + \frac{\zeta'(2)}{12\pi^2} \approx -.2815791335514.$$

The Hurwitz zeta function

All of these ψ_{-n} integrals are expressible via the first derivative of the Hurwitz zeta function $\zeta(s,a)$. Differentiating ∂s the reflection formula [Whittaker, 1927],

$$\frac{\partial}{\partial s}\frac{(2\pi)^s\zeta(1-s,a)}{\Gamma(s)} = -\sum_{n\geq 1}\frac{\pi\sin(s/2-2an)\pi + 2(\ln n)\cos(s/2-2an)\pi}{n^s},$$

which, for integer s, can be written

$$\zeta'(1-s,a) = B_s(a)\frac{\psi_0(s)-\ln 2\pi}{s} - (-)^s\frac{(s-1)!}{(2\pi)^{2s}}\frac{\partial^s}{\partial a^s}\sum_{n\geq 1}\frac{\pi\sin(2\pi an) - 2\ln(n)\cos(2\pi an)}{n^{2s}},$$

where B_s is the Bernoulli polynomial

$$B_s(a) = -s\zeta(1-s,a).$$

With this we eliminate the series from equation (ψ_{-2}),

$$\int_0^a \ln\Gamma(t)dt = a\frac{1-a+\ln 2\pi}{2} + \zeta'(-1,a) - \zeta'(-1),$$

and from its integration by parts,

$$\int_0^a t\ln\Gamma(t)dt = \frac{a}{24} + a^2\frac{1+2\ln 2\pi}{8} - \frac{a^3}{4} + a\zeta'(-1,a) - \frac{\zeta'(-2,a)}{2} + \frac{\zeta(3)}{8\pi^2},$$

etc.

We also note that ζ' provides a concise, analytic extension of the “higher factorials”, *e.g.*:

$$1^{1^s}2^{2^s}\dots n^{n^s} = e^{\zeta'(-s,n+1)-\zeta'(-s)},$$

$$1!2!\dots n! = n!^{n+1}e^{\zeta'(-1)-\zeta'(-1,n+1)} = e^{(n+1)\left(\zeta'(0,n+1)-\zeta'(0)\right)+\zeta'(-1)-\zeta'(-1,n+1)},$$

and reflection in the second argument provides polylogarithms, *e.g.*:

$$\mathrm{Li}_2(e^{2i\pi z}) = \pi^2 B_2(z) + 2i\pi\left(\zeta'(-1,z) - \zeta'(-1,1-z)\right).$$

With $z = 1/6$ and our earlier result for $\zeta'(-1,1/6)$,

$$\mathrm{Li}_2(e^{i\pi/3}) = \frac{\pi^2}{36} + \frac{i}{\sqrt{3}}\left(\frac{\psi_1(1/3)}{2} - \frac{\pi^2}{3}\right) \approx .2741556778080377 + 1.014941606409653i,$$

but this merely reminds us that we can express $\mathrm{Li}_k(e^{ip\pi/q})$ in terms of $\psi_{k-1}(j/q), 0 < j < q$, simply by grouping the terms of the Li series q at a time, leading to a rational function summand.

References:

Cohen, Henri, Leonard Lewin, and Don Zagier, *A Sixteenth-order Polylogarithm Ladder*, Experimental Mathematics, vol. 1 (1992), no. 1, p26.

Glaisher, James W. L., *Products and Series involving Prime Numbers Only* (Part II), Quarterly Journal of Pure and Applied Mathematics, vol XXVIII (1895), p4.

Gosper, R. Wm., *Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics*, Computers in Mathematics, (D. Chudnovsky & R. Jenks, eds.), Lecture Notes in Pure and Applied Mathematics, Vol 125 (1990), p282.

Knuth, D. E., *An evaluation of Porter's constant*, Computers and Mathematics, with Applications, v.2 (1976), pp137-139.

Liang, J. J. Y., and John Todd, *The Stieltjes Constants*, J. Res. Nat. Bur. Standards, vol 76 (1972), pp161-178.

Pitteway, M. L .V. and C. M. A. Castle, *On the subtractive version of Euclid's algorithm*, Bulletin of the Institute of Mathematics and its Applications, vol. 24 (1988), no. 1-2, pp17-21.

Whittaker, E. T., and G. N. Watson, Modern Analysis, Cambridge University Press, chap XIII (1927), p269.