

Notes on Binomial Coefficient Extension

W. Fred Lunnon, NUI Maynooth, Rep. of Ireland

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Abstract

Extension of the classical binomial coefficient $\binom{m}{n}$ from natural to integer arguments m, n is complicated by an unavoidable contradiction: although the elementary laws of recurrence $\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$ and symmetry $\binom{m}{n} = \binom{m}{m-n}$ hold for both natural arguments and their extension to real arguments via the Gamma function, it is impossible simultaneously to reconcile both laws with the definition in terms of Taylor series for natural n . This has misled some authors into assigning nonzero values to a third sector of the lattice where $n \leq m < 0$ — *tri-wedge* regime — rather than simply assigning zero where $n < 0$ — *bi-wedge* regime.

The article expounds the argument for condemning the tri-wedge option, on the grounds that it destroys the mathematical consistency which permits a coherent strategy to stating and proving theorems about binomial coefficients, while offering no concrete advantage for any other purpose.

In this connection, current computer algebra systems are found to perform poorly at validating binomial coefficient identities, a situation exacerbated by incomplete attempts to implement a tri-wedge regime. Finally, widely available purported graphical plots of the corresponding function of real arguments are discovered to be grossly incomplete.

1 Binomial Coefficients in the Plane

Initially the binomial coefficients $\binom{m}{n}$ arise for natural m, n only, as the number of distinct subsets with cardinal n of a set of distinct members with cardinal m — the 6 o'clock and 4 o'clock sectors of Figure 1. From the explicit formula

$$\binom{m}{n} = m(m-1) \dots (m-n+1)/n(n-1) \dots 1 \quad (1)$$

it is a small intuitive step to extend them to integer m and natural n — attaching the 2 o'clock sector. Consideration of coefficients in the power series expansion

$$(1+v)^m = \sum_n \binom{m}{n} v^n \quad (2)$$

suggests the further extension to real x and integer n

$$\binom{x}{n} = \begin{cases} x(x-1)\dots(x-n+1)/n! & \text{for } n \geq 0, \\ 0 & \text{for } n < 0; \end{cases} \quad (3)$$

in particular, a plausible definition for integer arguments now seems to be in place, albeit reliant on the somewhat debatable interpretation of equation (2) as a Laurent expansion incorporating negative exponents n , rather than more restricted Taylor-Maclaurin sum.

Naturally (or perhaps really) no mathematician is going to stop there. Further progress involves recasting formula (1) entirely in terms of factorials

$$\binom{m}{n} = m!/n!(m-n)! \quad (4)$$

— initially a backward step, valid again only for m, n natural. Euler overcomes the obstruction by extending the factorial function canonically to real arguments via the Gamma function $\Gamma(x)$, complete with irritating change of origin so that $k! = \Gamma(k+1)$ for natural k . We might now attempt to redefine $\binom{x}{y}$ as

$$\Delta(x, y) = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)} \quad (5)$$

over the whole real (x, y) -plane: see Figure 3 below, with $\binom{m}{n}$ values superposed as red balls.

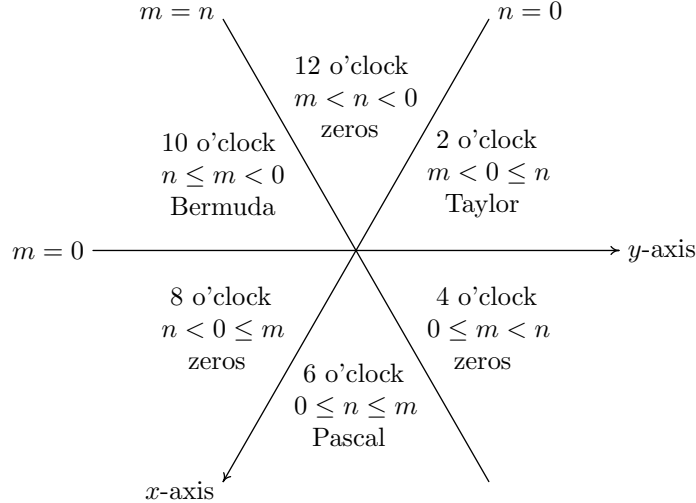


Figure 1: Symmetrically sheared sectors of $\binom{m}{n}$ plane

But there remains a complication, occasioned by simple poles of $\Gamma(z)$ at negative integer $z = -k$, around which for small h

$$\Gamma(-k + h) = \frac{(-1)^k}{k!} h^{-1} + \mathcal{O}(1) \quad (6)$$

[a straightforward consequence of the reflection formula (11) below].

Away from lattice points, a solitary pole in the denominator when $y = n < 0$ or $x - y = m - n < 0$ (4 and 8 o'clock sectors) plainly yields zero on the right-hand side of equation 5), and a solitary pole in the numerator when $x = m < 0$ yields infinity. At a lattice point where $x + 1 = m < n = y + 1 < 0$ (12 o'clock sector), the double pole in the denominator overcomes the single pole in the numerator, yielding zero. Finally only at lattice points $y = n \leq m = x < 0$ or $m = x < 0 \leq y$ (10 and 2 o'clock sectors) does $\Delta(x, y)$ remain to be determined, and with it $\binom{m}{n}$.

A plausible alternative strategy to determine the 10 o'clock sector involves applying established relations between binomial coefficients to extend the table, such as the elementary *recurrence*

Assertion 1. *For integer m and natural n ,*

$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}; \quad (7)$$

and *symmetry*

Assertion 2. *For natural m, n ,*

$$\binom{m}{n} = \binom{m}{m-n}; \quad (8)$$

both following easily from equation (1). In fact, both properties hold also for $\Delta(x, y)$ everywhere it is univalent — everywhere that is, excepting $\Delta(m, n)$ at 10 and 2 o'clock.

Extending via recurrence (7) is conveniently consistent with Taylor-Laurent series, equations (2), (3): whence the *bi-wedge* regime

Definition 3. *For integer m, n with $n \leq m < 0$,*

$$\binom{m}{n} = 0 \quad (!!); \quad (9)$$

Extending via symmetry (8) instead prefers to fill the 10 o'clock sector with the 2 o'clock sector, but reflected in the vertical symmetry axis line $m = 2n$: whence the *tri-wedge* regime

Definition 4. *For integer m, n with $n \leq m < 0$,*

$$\binom{m}{n} = \binom{m}{m-n} = (-1)^{m-n} \binom{-n-1}{m-n} \quad (??) \quad (10)$$

Beguilingly, this does also agree with equation (7) nearly everywhere, the sole exception lurking at the origin $(m, n) = (0, 0)$ where recurrence would require that $1 + 1 = 1$. Faced with avoiding disagreement over an entire plane sector, at the expense of single measly point — hey, no contest, right?

[illegible]

Figure 2: $\binom{m}{n}$ bi-wedge (*hooray!*) versus tri-wedge (*boo!*) regime

At least, that's how some authors appear to have felt, prior to embarking on a hunt for fallacies of varying degrees of implausibility in support of their case. For example the perpetrator of [4], having blithely set $x = 0$ in the Euler reflection formula

$$\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x) \quad (11)$$

(elsewhere valid: see [7] §12.14), subsequently accounts for the discrepancy at the origin with the illuminating observation that “This is a consequence of the fact that addition and division are non-associative for infinitely small numbers.” Somewhat less boldly [6], having quoted equation (6) more or less correctly, goes on to assume with neither notification nor justification that singularities of $\Delta(x, y)$ are to be approached along $dy/dx = 1$, inadvertently emptying the 2 o’clock sector in the process — see Assertions 5, 6 below.

From the opposing corner Knuth *et al* tell it like it is, implicitly on the grounds of mathematical coherence explored in Section 3 —

“So the equation ‘ $\binom{-1}{n} = \binom{-1}{-1-n}$ ’ is always false! The symmetry identity fails for all other negative integers m , too. But unfortunately its all too easy to forget this restriction, since the expression in the upper index is sometimes negative only for obscure (but legal) values of its variables. Everyone who’s manipulated binomial coefficients much has fallen into this trap at least three times.”

([2] §5.1 p.156).

2 Surface Topography

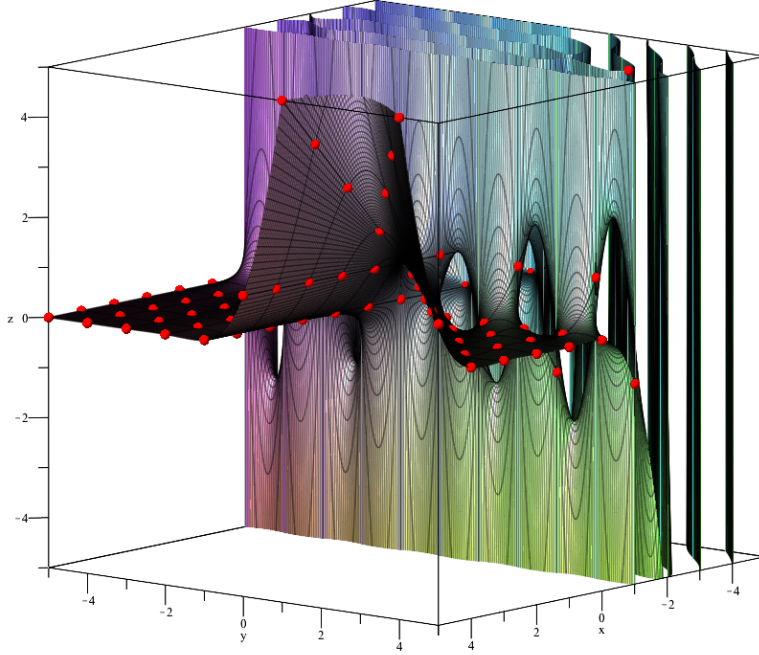


Figure 3: $\Delta(x, y)$ viewed from above the carpet at 4 o'clock

At this juncture — protesting vociferously at vigorous encouragement from Bill Gosper, which is hereby belatedly acknowledged — the humble seeker after truth is eventually obliged investigate the actual behaviour of $\Delta(x, y)$ in the neighbourhood of a lattice point (m, n) in the 10 o'clock sector. Translating the origin to (m, n) , equation (5) becomes

$$z = \frac{\Gamma(m+1+x')}{\Gamma(n+y'+1)\Gamma(m-n+x'-y'+1)}$$

where $(x', y') = (r \cos \theta, r \sin \theta)$ and r is small; expanding z in terms of r via equation (6),

$$z = \binom{m}{m-n} \tan \theta + \mathcal{O}(r);$$

and letting $r \rightarrow 0$,

Assertion 5. *For integer $n \leq m < 0$, as (x, y) approaches (m, n) along gradient $dy/dx = s$, the right-hand side in equation (5) approaches limit $\binom{m}{m-n}s$.*

As an immediate corollary, the surface $z = \Delta(x, y)$ in the neighbourhood of such a lattice point is a hyperbolic paraboloid, tilted so that one generator of zero curvature is the vertical line through (m, n) : see the pale lines in Figure 4. By varying the approach gradient s , the limiting value of z at the lattice point may be specified arbitrarily!

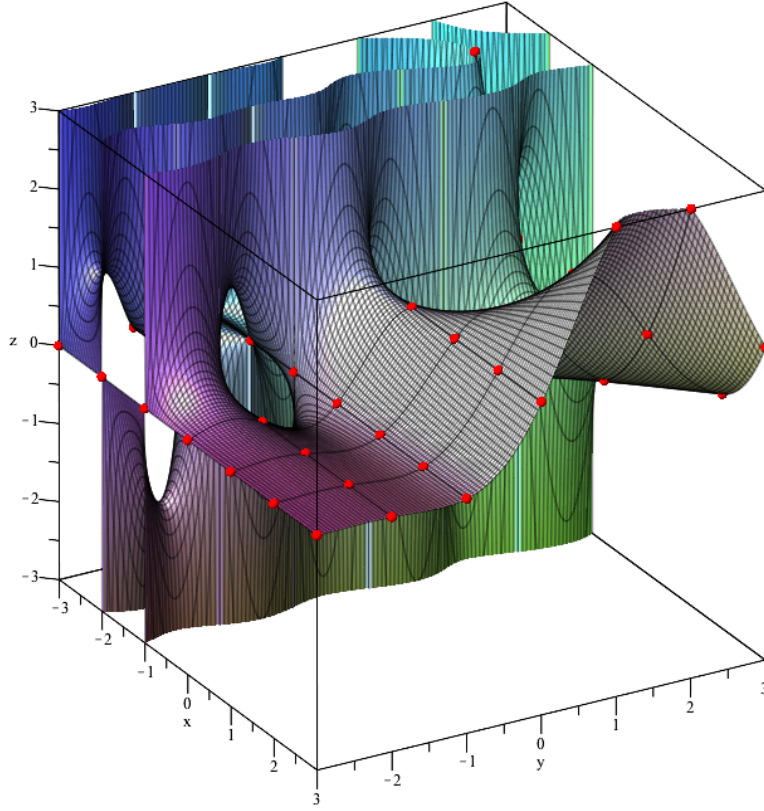


Figure 4: $\Delta(x, y)$ close-up, from above the carpet at 8 o'clock

Which may not suggest useful progress, but patience: in the 2 o'clock sector it transpires in a similar fashion that

Assertion 6. *For integer $m < 0 \leq n$, as (x, y) approaches (m, n) along gradient $dy/dx = s$, the right-hand side in equation (5) approaches limit $\binom{m}{n}(1 - s)$.*

The surface is locally again hyperbolic paraboloid, but rotated $\pi/4$ around its vertical generator line relative to those at 10 o'clock. Specifying Taylor series values at 2 o'clock requires $s = 0$, which in turn entails zeros at 10 o'clock and confirms the bi-wedge definition (9).

And just to complete our singularity zoo, at 12 o'clock where $m < n < 0$ we find $z = \mathcal{O}(r)$; there is little doubt about the limit being zero in this region, despite the continuing presence of a vertical generator line, since the surface here is locally a cone with pinch vertex at the lattice point and on the $z = 0$ plane: see Figure 5. The earnestly sought universal definition emerges in the final form

Assertion 7.

$$\binom{x}{y} = \lim_{dx \rightarrow 0} \frac{\Gamma(x+1+dx)}{\Gamma(y+1)\Gamma(x-y+1+dx)} \quad (12)$$

[From a formal standpoint, and ignoring the Taylor series constraint, any consistent choice of s becomes possible alongside $s = 0$. For example $s = 1$ yields the 10 o'clock sector of the tri-wedge regime (10), at the expense of clearing the 2 o'clock sector to zeros instead — demonstrating the proverbial infeasibility of retaining one's cake while simultaneously ingesting it. $s = 1/2$ on the other hand restores symmetry, but halves the value at all lattice points for $m < 0$ — one may notwithstanding have half one's cake and eat the other half. Approaching along the y -axis with $s = \infty$ results in ∞ at every lattice point for $m < 0$, including the 2 o'clock sector — it is possible to have (far) too much of a good thing ...]

The behaviour of the surface in the neighbourhood of lines $x = m$ between lattice points for integer $m < 0$ is also worth recording, not least on account of the indigestion it causes computer plotting utilities. As a function of y alone, there is a simple pole where the surface crosses $x = m$; the y, z cross-section is therefore locally a hyperbola with one asymptote vertical. Translating along the x -direction, the gap between branches traces a sloping 'nostril' breach in the surface, bounded on either side by singularities at neighbouring lattice points: see Figure 3.

For $x < 0$ therefore the surface forms a sequence of asymptotically vertical curtains at integer x , each punctuated near the plane $z = 0$ by a series of nostrils, between integer y , and sloping in alternate directions. In the 12 o'clock sector however the nostrils close up to form lumpy connected platforms of pinch-points between curtains: see Figure 5. [In a finite plot, curtains near the extremities appear to fold around and interconnect, in an unpredictable fashion and contradicting the symmetry equation (8) which should hold (almost) everywhere. This is an illusion, caused by a combination of the shear imposed by rectangular axes, together with rapid growth in nostril height with distance from the origin.]

For $x > 0$ there is in contrast simply a smooth 'carpet' of near-zeros laid across 4 and 8 o'clock sectors of the xy -plane, disrupted by a trapped 'shark' projecting obtrusively from the 6 o'clock sector. Seen from this carpet, nostrils

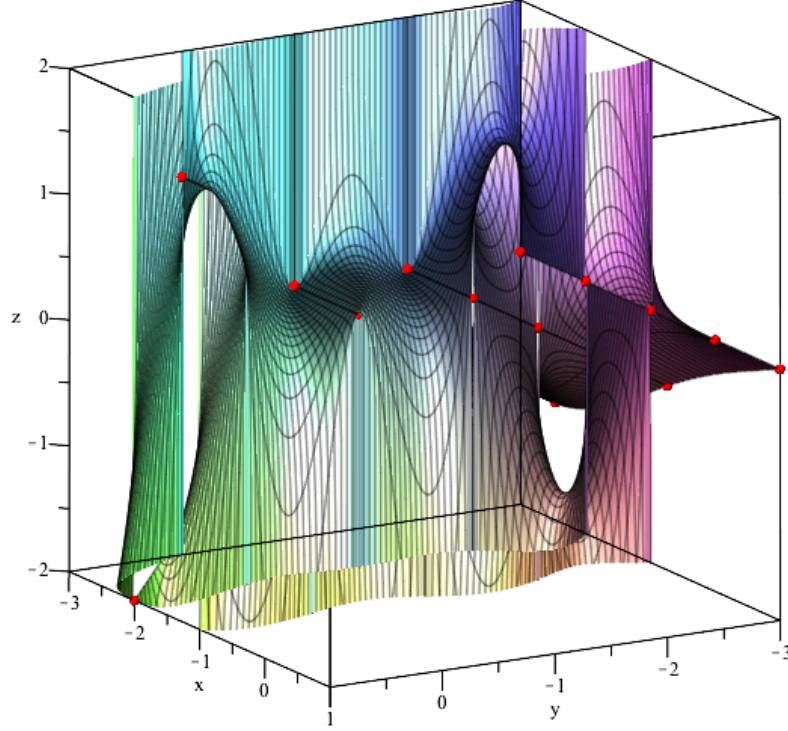


Figure 5: $\Delta(x, y)$ conical platform, from below at 12 o'clock

appear to align in connected ‘tunnels’ penetrating the curtains, to finally reconnect in pairs at the platforms to the rear.

This involved topography is responsible for some spectacularly unsuccessful previous attempts to plot the surface, along with the closely related classical Beta function

$$B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y) = 1/(x + y - 1) \Delta(x - 1, y - 1). \quad (13)$$

Compare for example Figure 6 with the graphic travesty currently gracing [5] and elsewhere: the latter’s sole noteworthy feature is a lattice of vertical planar artifacts, resulting from attempting to plot across lines of discontinuity, and completely obliterating any structural detail of the actual surface. Even when such planes have been circumvented, the plot is then apt to succumb in their stead to spurious open fissures, where it should show vertical lines arising from

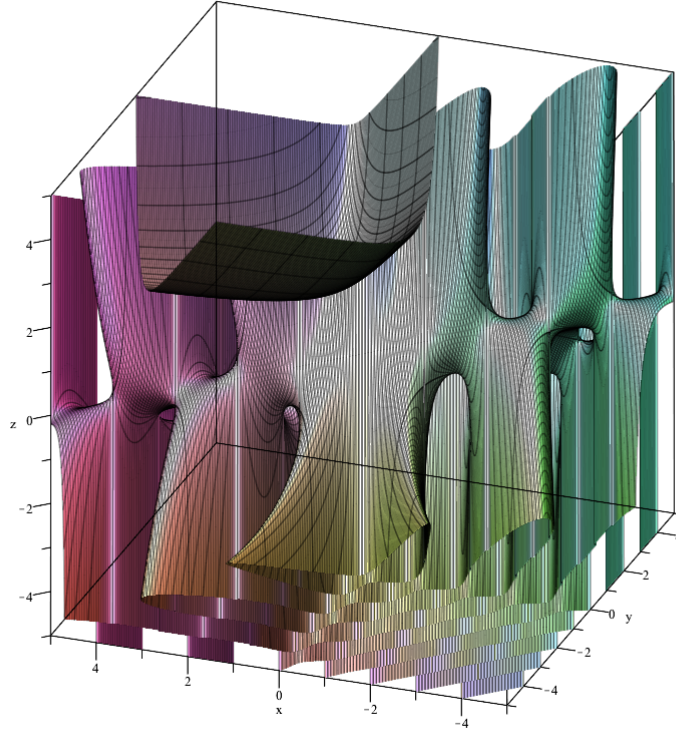


Figure 6: Beta function from below at 4 o'clock

singularities at lattice points.

Mysteriously, the Maple implicit plotting option — which might be expected to avoid these artifacts — generates more of them than the standard explicit function plot. The only cure available for such woes is to decompose the argument plane into a tiling of squares in the interiors of which the function remains both single-valued and well-behaved, then re-assemble those into the final surface.

3 Automatic Theorems

Some computer algebra systems, in particular Maple (version 17) and Mathematica (version 9), may be induced to attempt to prove an identity involving binomial coefficients, by reformulating it in terms of hypergeometric and Gamma functions which can then (on occasion) be resolved to zero. This procedure proves fraught with complications.

In the first place, the identity must be valid for all real arguments, not just integers: for example, any summation range involving a parameter must be extendable to infinity — in turn potentially involving negative arguments previously avoided.

Secondly, the current versions of both CAS above adopt the deprecated tri-wedge regime (10), with the consequence that any result thus established is liable to be either invalid or internally inconsistent if involving negative arguments anywhere. At time of writing, Mathematica (qua Wolfram Alpha) boldly asserts that equation (7) in the form

$$(\text{Binomial}[m-1, n-1] + \text{Binomial}[m-1, n] = \text{Binomial}[m, n])$$

is “True” for variable m, n , whilst at the origin

$$(\text{Binomial}[m-1, n-1] + \text{Binomial}[m-1, n] = \text{Binomial}[m, n]) \\ /. \{m \rightarrow 0, n \rightarrow 0\}$$

is “False” — contradicting itself and the documentation. Maple just flatly declines to reduce either of equations (7) or (8), carrying prudence to what might seem excessive lengths.

Thirdly, even if the glitch above were to be corrected, there would remain the more serious issue of $\Delta(x, y)$ singularities in equation (5). To evaluate an expression for all integer arguments, the target value must be approached along a consistent tangent $s = 0$: for example, attacking equation (8) in Maple via

```
Delta := proc(x, y)
  GAMMA(x+1)/GAMMA(y+1)/GAMMA(x-y+1)
end;
func := proc(n, k)
  limit(limit(Delta(n+e, k+d) = Delta(n+e, n-k+d), d = 0), e = 0)
end;
func(1,1); func(-1,-1);
```

delivers desired bi-wedge results, albeit it in characteristically perverse form $1 = 1$ and $0 = 1$. [A simple implementation of equation (3) would achieve the same result with less effort.]

But while this strategy works for individual values, once variables are involved it becomes impractical: equation (7) in the general form

```
n,k,e,d := 'n','k','e','d';
func := limit(limit(
  Delta(n+e, k+d) - Delta(n-1+e, k+d) - Delta(n-1+e, k-1+d),
  d = 0), e = 0);
simplify(func);
```

consumes 3.5 minutes and 400 Mbytes in producing a rational quotient of Gamma functions which formally simplifies to zero. However closer inspection reveals that removable singularities render the raw result undefined outside the 6 o'clock sector, the original Pascal triangle!

Despite these initially discouraging experiments, a glimmer of optimism may be extracted from tackling the ‘hockey stick’ identity: in correct bi-wedge form for all integer m, n

$$\binom{m}{n} = \sum_{j \geq 0} \binom{m-1-j}{n-j}, \quad (14)$$

($0 \leq j \leq n$ in practice). Maple immediately evaluates the sum to

$$\binom{m-1}{n} m / (m-n)$$

which reduces to the left-hand side, unless $m = n$ (when the effective range is just $j = n$). This essentially successful outcome is achieved only at the expense of internal consistency, since under the tri-wedge regime equation (14) fails throughout the 10 o’clock sector.

A fourth hazard concerns summation conventions: Maple distinguishes explicitly between iterated addition and formal summation, unlike Mathematica. Over an interval with negative length it is not obvious whether such a sum is defined to be merely empty (Maple `add(...)`), directed like an integral (Mathematica `Sum[...]` with luck), or unpredictably neither (Maple `sum(...)`, Mathematica often).

```
Sum[j, {j,0,k}] /. {k -> -2} (* result = 1, directed *)
Sum[j, {j,0,-2}] (* result = -2, kludged *)
```

Such conundrums are avoided when the bounds can be kept constant.

Finally, while the bi-wedge regime is successful in streamlining identities into a suitable form for automatic processing, it must regretfully be concluded that few useful, reliable results are at present within reach through this avenue.

4 Sample Identities

This section discusses further binomial coefficient identities of interest, to illustrate how a clean formulation with constant summation bounds and universal parameter range is achievable under the bi-wedge regime.

The Vandermonde convolution: for all integer m, n, k

$$\binom{m}{k} = \sum_j \binom{m-n}{k-j} \binom{n}{j} \quad (15)$$

($0 \leq j \leq k$ in practice). With bounds $0 \leq j \leq \infty$, Maple reduces this successfully in the region $n+k < m$ & $m \geq 0$. The tri-wedge regime anyway scuppers things for $k < m < 0$, but it is unclear why n has not also been eliminated: it was for $0 \leq j \leq k$, but the summation then failed to progress further.

The ballot numbers (not to be confused with the Beta function above) are defined by

$$\begin{aligned} B(n, m) &= \binom{n+m-1}{m} - \binom{n+m-1}{m-1} \\ &= \binom{n+m}{m} (n-m)/(n+m) \quad \text{if } n+m \neq 0; \end{aligned} \quad (16)$$

they satisfy the convolution for all m, n

$$B(m, n) = \sum_j B(m-1-j, n-j) B(j+1, j) \quad (17)$$

($0 \leq j \leq n$ in practice).

Formulation as a numerical Gamma function (bi-wedge) limit

```
func := proc(m, n) local d, e;
  limit(limit(
    (Delta(m+k-1+e, n+d) - Delta(m+n-1+e, n-1+d)) - add(
      (Delta(m+n-2*j-2+e, n-j+d) - Delta(m+n-2*j-2+e, n-j-1+d))
      * (Delta(j+j+e, j+d) - Delta(j+j+e, j-1+d)), j = 1..n),
    d = 0), e = 0) end;
  1 := 7; matrix([seq([seq(func(m, n), n = -1..1)], m = -1..1)]);
```

delivers a reassuring bank of zeros. However, the general version

```
m, n, e, d := 'm', 'n', 'e', 'd';
limit(limit(
  (Delta(m+n-1+e, k+d) - Delta(m+n-1+e, n-1+d)) -
  sum((Delta(m+n-2*j-2+e, n-j+d) - Delta(m+n-2*j-2+e, n-j-1+d))
    * (Delta(j+j, j+d) - Delta(j+j+e, j-1+d)), j = 1..n),
  d = 0), e = 0);
```

merely gallops up the store and times out; relaxing the bounds to infinity has no effect. In fact, Maple is unable to evaluate even the single limit with respect to d ; and omitting both limits in desperation stumbles upon the curious fact that

```
sum(Delta(2*n-2*j, n-j-1), j = n+1..n+1);
```

evaluates neither to 0 (bi-wedge consistent) nor to 1 (tri-wedge), but to $1/2$.

Catalan numbers are a special case of ballot numbers

$$\begin{aligned} C(n) &= B(n+1, n) = \binom{2n}{n} - \binom{2n}{n-1} \\ &= B(n+1, n-1) \quad \text{if } n \neq 0 \\ &= \binom{2n}{n} / (n+1) \quad \text{if } n \neq -1, \end{aligned} \quad (18)$$

with convolution following from equation (17) for $n \neq 0$

$$C(n) = \sum_j C(n-1-j)C(j) \quad (19)$$

($0 \leq j \leq n-1$ in practice).

Given the sum in the form

```
func := sum(
  (binomial(2*j-2, j-1) - binomial(2*j-2, j-2)) *
  (binomial(2*n-2*j, n-j) - binomial(2*n-2*j, n-j-1)),
  j = 1..n);
simplify(func);
```

Maple quickly reduces this to (essentially)

$$4^n \Gamma(n+1/2) / (\sqrt{\pi} \Gamma(n) n(n+1))$$

which it can then simplify to the left-hand side. But this example illustrates the importance of inspecting the raw output of the summation, which here has a removable singularity at $n = 0$ where the identity fails, and another at $n = -1$ where it would also fail under the tri-wedge regime.

Finally, consider a less well-known identity ascribed to Catalan and presented in [6] essentially in the form

$$\sum_{0 \leq k \leq n} \binom{p-k-1}{p-n-1} \binom{q+k}{m} = \sum_{0 \leq k \leq m} \binom{q-k-1}{q-m-1} \binom{p+k}{n} \quad ?$$

Under tri-wedge as advocated there, this would fail at $m, n, p, q = 0, -1, -2, 0$. Under bi-wedge it fails at $0, 0, -1, 0$; but the inequivalent variation

$$\sum_{k \geq 0} \binom{p-k-1}{n-k} \binom{q+k}{m} = \sum_{k \geq 0} \binom{q-k-1}{m-k} \binom{p+k}{n} \quad (20)$$

holds for all integer m, n, p, q . A straightforward if tedious proof proceeds via multiple induction on several subregions using Assertion 7, noting that when $p = q = 0$ both sums equal δ_{mn} for $m, n \geq 0$ and 0 otherwise.

The further claim advanced in [6] — that both sums evaluate to the single binomial coefficient $\binom{p+q}{n-m+q}$ for natural m, n — is fallacious unless

$$m = 0 \quad \text{or} \quad n = 0 \quad \text{or} \quad -m \leq p \leq m \quad \text{or} \quad -n \leq q \leq n.$$

Outside such regions of the parameter lattice, the numerical evidence indicates plainly that the sums cease to be expressible by any single binomial coefficient. It might be worthwhile to investigate this tantalising inconsistency further!

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