§19.2 Our first proof of Jacobi's *identica abstrusa*

We start with the trivial 2–dissection

$$\phi(q) = \phi(q^4) + 2q\psi(q^8). \tag{19.2.1}$$

If in (19.2.1) we put $-q$ for q, we obtain

$$\phi(-q) = \phi(q^4) - 2q\psi(q^8). \tag{19.2.2}$$

If we multiply (19.2.1) by (19.2.2) and use (1.6.7), we obtain

$$\phi(-q^2)^2 = \phi(q^4)^2 - 4q^2\psi(q^8)^2. \tag{19.2.3}$$

If in (19.2.3) we replace q^2 by q, we obtain

$$\phi(-q)^2 = \phi(q^2)^2 - 4q\psi(q^4)^2. \tag{19.2.4}$$

If in (19.2.4) we put $-q$ for q, we obtain

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{19.2.5}$$

If we square (19.2.5), take the odd part and use (1.6.6) with q^2 for q, we obtain

$$\mathbf{O}\left(\phi(q)^4\right) = 8q\phi(q^2)^2\psi(q^4)^2 = 8q\psi(q^2)^4, \tag{19.2.6}$$

where $\mathbf{O}\left(f(q)\right) = \dfrac{f(q) - f(-q)}{2}$ is the **O**dd part of $f(q)$.

That is,

$$\mathbf{O}\left((-q,-q,q^2;q^2)_\infty^4\right) = 8q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}. \tag{19.2.7}$$

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If we divide (19.2.7) by $(q^2;q^2)_\infty^4$, we find

$$\mathbf{O}\left((-q;q^2)_\infty^8\right) = 8q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^8} = 8q(-q^2;q^2)_\infty^8, \tag{19.2.8}$$

which is (19.1.1).

§19.3 Our second proof of *identica abstrusa*

Let

$$\rho(q) = \frac{\phi(q)}{\psi(q^2)}. \tag{19.3.1}$$

We have, by (19.2.5) and (1.6.6) with q^2 for q,

$$\rho(q)^2 = \frac{\phi(q)^2}{\psi(q^2)^2} = \frac{\phi(q^2)^2 + 4q\psi(q^4)^2}{\phi(q^2)\psi(q^4)} = \frac{\phi(q^2)}{\psi(q^4)} + 4q\frac{\psi(q^4)}{\phi(q^2)} = \rho(q^2) + \frac{4q}{\rho(q^2)}. \tag{19.3.2}$$

If we square (19.3.2) and take odd parts, we find

$$\mathbf{O}\left(\rho(q)^4\right) = 8q, \tag{19.3.3}$$

which is (19.2.6). Now complete the proof as in §19.2.

Note that if we put $-q$ for q in (19.3.3), we obtain

$$\mathbf{O}\left(\rho(-q)^4\right) = -8q, \tag{19.3.4}$$

$$\mathbf{O}\left(\left(\frac{\phi(-q)}{\psi(q^2)}\right)^4\right) = -8q, \tag{19.3.5}$$

$$\mathbf{O}\left(\left(\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\Big/\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}\right)^4\right) = -8q, \tag{19.3.6}$$

$$\mathbf{O}\left(\left(\frac{(q;q)_\infty}{(q^4;q^4)_\infty}\right)^8\right) = -8q, \tag{19.3.7}$$

or, finally,

$$\mathbf{O}\left(\prod_{4\nmid n}(1-q^n)^8\right) = -8q. \tag{19.3.8}$$

(See the front cover!)

To bring home how stunning this is, observe that

$$\prod_{4\nmid n}(1-q^n)^8 = 1 - 8q + 20q^2 - 62q^4 + 216q^6 - 641q^8 + 1636q^{10} - 3778q^{12} + 8248q^{14} \\ - 17277q^{16} + 34664q^{18} - \cdots .$$

Furthermore, if we define

$$\eta(q) = q^{\frac{1}{24}}E(q)$$

and H is the huffing operator modulo 2, that is,

$$H\left(\sum a_n q^n\right) = \sum a_{2n}q^{2n},$$

then *identica abstrusa* can be written

$$H\left(\left(\frac{\eta(q)}{\eta(q^4)}\right)^8\right) = -8.$$