Jacobi's *aequatio identica satis abstrusa*

Let

$$\phi(q)=\sum_{-\infty}^{\infty} q^{n^2}, \qquad \psi(q)=\sum_{n\ge 0} q^{(n^2+n)/2}.$$

Then it is easy to show, using Jacobi's triple product identity, that

$$\phi(q)^2=\phi(q^2)^2+4q\psi(q^4)^2,$$

and

$$\psi(q)^2=\phi(q)\psi(q^2).$$

Now let

$$\rho(q)=\frac{\phi(q)}{\psi(q^2)}.$$

Then

$$\rho(q)^2=\frac{\phi(q)^2}{\psi(q^2)^2}=\frac{\phi(q^2)^2+4q\psi(q^4)^2}{\phi(q^2)\psi(q^4)}=\frac{\phi(q^2)}{\psi(q^4)}+4q\frac{\psi(q^4)}{\phi(q^2)}=\rho(q^2)+\frac{4q}{\rho(q^2)}.$$

If we square this and extract odd parts, we obtain

$$\mathrm{O}\left(\rho(q)^4\right)=8q,$$

which can be written

$$\mathrm{O}\left(\phi(q)^4\right)=8q\psi(q^2)^4,$$

$$\phi(q)^4-\phi(-q)^4=16q\psi(q^2)^4,$$

$$(-q,-q,q^2;q^2)_\infty^4-(q,q,q^2;q^2)_\infty^4=16q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4},$$

$$(-q;q^2)_\infty^8-(q;q^2)_\infty^8=16q\frac{(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^8},$$

or, finally,

$$\prod_{n\geq 1}(1+q^{2n-1})^8 - \prod_{n\geq 1}(1-q^{2n-1})^8 = 16q\prod_{n\geq 1}(1+q^{2n})^8.$$

The first three lines constitute the proof, given the lead–up of the last five lines, as in Chapter 19 of my upcoming book, “The power of q”.