Motion of a Rigid Body

Santa Cruz CA

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Introduction

The exact description of the torque-free motion of a rigid body whose principal moments of inertia are all different is a nontrivial, yet solvable, exercise in mathematical physics. The present report discusses the solution within classical mechanics. The author plans to eventually expand the presentation to include the quantum mechanical problem.

The well known solution to this problem as reviewed here is based upon that of E. T. Whittaker [1].

The reader is assumed to possess an understanding of the basic physical concepts: moments of inertia, angular velocity, angular momentum and its conservation, rotational energy and its conservation, and the relationships between these quantities. A detailed understanding also is presumed for the mathematical theory of elliptic functions and theta functions, and their notational conventions, as can be found in the classic work of E. T. Whittaker and G. N. Watson [2].

Basic Mechanics

Rotating Coordinates

The orientation of a rotating body can be specified in a variety of ways. For the moment, we shall use an orthogonal matrix R. By $\mathbf{x}$ we denote a vector in a space-fixed coordinate system. A dual interpretation is intended: $\mathbf{x}$ can denote the set of coordinates with respect to the fixed basis as well as the abstract, coordinate-free, vector itself. By $\mathbf{x}_B$ we denote the coordinates of $\mathbf{x}$ with respect to a basis attached to and moving with the rotating body. In terms of coordinates, $\mathbf{x}_B = R\,\mathbf{x}$, and $\mathbf{x} = R^{-1}\mathbf{x}_B$.

By differentiating $R\,R^{-1} = 1$ with respect to time, it follows that

$$\frac{d}{dt}R^{-1} = -R^{-1}\frac{dR}{dt}R^{-1}.$$

Then

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \left(\frac{d}{dt}R^{-1}\right)\mathbf{x}_B + R^{-1}\frac{d\mathbf{x}_B}{dt}\\ &= -R^{-1}\frac{dR}{dt}R^{-1}\mathbf{x}_B + R^{-1}\frac{d\mathbf{x}_B}{dt}\\ &= -R^{-1}\frac{dR}{dt}\mathbf{x} + R^{-1}\frac{d\mathbf{x}_B}{dt}.\end{aligned}$$

Now, if the vector $\mathbf{x}$ happens to be body-fixed, so that $d\mathbf{x}_B / dt = 0$, we know from the definition of angular velocity $\boldsymbol{\omega}$ that

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\omega} \times \mathbf{x} \, .$$

It follows by comparison that for a general vector $\mathbf{x}$,

$$-R^{-1}\frac{dR}{dt}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x} \, .$$

We thus have

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{\omega} \times \mathbf{x} + R^{-1}\frac{d\mathbf{x}_B}{dt} \, ,$$

which in body coordinates becomes

$$\left(\frac{d\mathbf{x}}{dt}\right)_B = \boldsymbol{\omega}_B \times \mathbf{x}_B + \frac{d\mathbf{x}_B}{dt} \, .$$

Also,

$$\boldsymbol{\omega}_B \times \mathbf{x}_B = (\boldsymbol{\omega} \times \mathbf{x})_B = -\frac{dR}{dt}\mathbf{x} = -\frac{dR}{dt}R^{-1}\mathbf{x}_B \, .$$

If $\boldsymbol{\omega}_B = (\omega_1, \omega_2, \omega_3)$, then this last equation states that

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = -\frac{dR}{dt}R^{-1} \, .$$

Moment of Inertia

The eigenvalues I_1, I_2, I_3 of the inertia tensor are the *principal moments of inertia*, and we may order them so that $0 < I_1 < I_2 < I_3$. This tensor, being symmetric, can be diagonalized by a suitable choice of orthogonal coordinates, which are attached to the rotating body. Without loss of generality we may assume the inertia is I_1 about the x axis, I_2 about the y axis, and I_3 about the z axis, and that xyz form a right-handed system.

For a body whose density is everywhere nonnegative, $I_3 \le I_1 + I_2$.

To see this, let ρ be the density, and let

$$a = \int \rho \, x^2 dV, \quad b = \int \rho \, y^2 dV, \quad c = \int \rho \, z^2 dV \, .$$

Then $I_1 = b + c$, $I_2 = c + a$, $I_3 = a + b$, and so $I_1 + I_2 = a + b + 2c \ge a + b = I_3$.

Conservation of Energy and Angular Momentum

Let $\boldsymbol{\omega}$ be the angular velocity of the body; let $\mathbf{L}$ be its angular momentum; and let E be its rotational energy. The conservation laws state that

$$\frac{d\mathbf{L}}{dt} = 0, \quad \frac{dE}{dt} = 0 .$$

When resolved in body coordinates,

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3,$$

$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}. ,$$

A qualitative picture of the motion is now possible. Although L_1, L_2, and L_3 are not constant, being components with respect to a rotating basis,

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L_1^2 + L_2^2 + L_3^2 = L^2$$

is constant. Then in (L_1, L_2, L_3) space, the motion must lie on the intersection of the angular momentum sphere and the energy ellipsoid. It is clear that

$$\frac{L^2}{I_3} \le 2E \le \frac{L^2}{I_1} .$$

When E is slightly larger than its minimum value, $\boldsymbol{\omega}_B$ and $\mathbf{L}_B$ describe small curves surrounding the z axis of the body, and a similar statement is true when E is slightly smaller than its maximum value. However, when $2E$ is near L^2 / I_2, these vectors will swing across between the $+y$ and $-y$ directions. In fact, when $2E = L^2 / I_2$, the sphere-ellipsoid intersection consists of two ellipses which intersect on the y axis. It will be shown later that in this interesting case, the actual locus consists of one sweep of one of the four semiellipses joining $+y$ with $-y$.

If the body is not truly rigid, then time varying stresses and strains will dissipate energy. The energy ellipsoid shrinks until it is tangent to the angular momentum sphere along the z axis. The body then possesses the minimum possible energy for the given angular momentum, and is rotating about its axis of maximum inertia.

For the remainder of this paper, we will assume for the purposes of normalization that

$$\frac{L^2}{I_3} \le 2E \le \frac{L^2}{I_2} .$$

In this case, ω_3 always has the same sign, which by choice of direction of the z axis can be assumed to be positive, while ω_1 and ω_2 oscillate about 0. The alternative case can be obtained either by analytic continuation or by repeating the calculation under the alternative assumption.

Solution of the Rigid Body Problem

Angular Velocity in Body Coordinates

From section ,

$$\left(\frac{d\mathbf{L}}{dt}\right)_B = 0 = \boldsymbol{\omega}_B \times \mathbf{L}_B + \frac{d\mathbf{L}_B}{dt}.$$

Thus we obtain the equations of motion for the angular velocity components in body coordinates.

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3, \\ I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_3\omega_1, \\ I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2. \end{aligned}$$

It is easily verified directly that these equations imply the conservation of E and L^2 as given in section .

Comparison of the equations of motion above with the derivatives of the Jacobi elliptic functions:

$$\begin{aligned} \frac{d}{dx}\mathrm{sn}(x,k) &= \mathrm{cn}(x,k)\,\mathrm{dn}(x,k), \\ \frac{d}{dx}\mathrm{cn}(x,k) &= -\mathrm{sn}(x,k)\,\mathrm{dn}(x,k), \\ \frac{d}{dx}\mathrm{dn}(x,k) &= -k^2\mathrm{sn}(x,k)\,\mathrm{cn}(x,k), \end{aligned}$$

suggests trying solutions

$$\begin{aligned} \omega_1 &= A\,\mathrm{cn}(\lambda(t-t_0),k), \\ \omega_2 &= B\,\mathrm{sn}(\lambda(t-t_0),k), \\ \omega_3 &= C\,\mathrm{dn}(\lambda(t-t_0),k). \end{aligned}$$

Assigning the dn function to ω_3 assures that ω_3 always has the same sign, and the convention $\omega_3 > 0$ is assured by taking $C > 0$.

The energy equation becomes (suppressing k and t_0)

$$\begin{aligned} 2E &= I_1A^2\,\mathrm{cn}^2\,\lambda t + I_2B^2\,\mathrm{sn}^2\,\lambda t + I_3C^2\,\mathrm{dn}^2\,\lambda t \\ &= I_1A^2(1-\mathrm{sn}^2\,\lambda t) + I_2B^2\,\mathrm{sn}^2\,\lambda t + I_3C^2(1-k^2\,\mathrm{sn}^2\,\lambda t) \\ &= (I_1A^2 + I_3C^2) + (-I_1A^2 + I_2B^2 - I_3C^2k^2)\,\mathrm{sn}^2\,\lambda t, \end{aligned}$$

and similarly the angular momentum equation becomes

$$\begin{aligned} L^2 &= I_1^2A^2\,\mathrm{cn}^2\,\lambda t + I_2^2B^2\,\mathrm{sn}^2\,\lambda t + I_3^2C^2\,\mathrm{dn}^2\,\lambda t \\ &= (I_1^2A^2 + I_3^2C^2) + (-I_1^2A^2 + I_2^2B^2 - I_3^2C^2k^2)\,\mathrm{sn}^2\,\lambda t. \end{aligned}$$

We now have four equations from which A^2, B^2, C^2, and k^2 can be determined.

$$
\begin{aligned}
I_1 A^2 + I_3 C^2 &= 2E, \\
I_1^2 A^2 + I_3^2 C^2 &= L^2, \\
I_1 A^2 - I_2 B^2 + I_3 C^2 k^2 &= 0, \\
I_1^2 A^2 - I_2^2 B^2 + I_3^2 C^2 k^2 &= 0.
\end{aligned}
$$

$$
\begin{aligned}
k^2 &= \frac{I_2 - I_1}{I_3 - I_2} \frac{2 I_3 E - L^2}{L^2 - 2 I_1 E}, \\
A^2 &= \frac{2 I_3 E - L^2}{I_1 (I_3 - I_1)}, \\
B^2 &= \frac{2 I_3 E - L^2}{I_2 (I_3 - I_2)}, \\
C^2 &= \frac{L^2 - 2 I_1 E}{I_3 (I_3 - I_1)}.
\end{aligned}
$$

We also have for $k'^2 = 1 - k^2$,

$$
k'^2 = \frac{I_3 - I_1}{I_3 - I_2} \frac{L^2 - 2 I_2 E}{L^2 - 2 I_1 E}.
$$

Any one of the three equations of motion for the angular velocity components now determines that

$$
\lambda^2 = \frac{(I_3 - I_2)(L^2 - 2 I_1 E)}{I_1 I_2 I_3},
$$

and also that the sign of λ must be the same as the sign of AB. We are not free to make both A and B have the same sign because we cannot reverse the direction of the x or y axis alone without destroying the right-handedness of the coordinates, having already chosen the direction of the z axis.

Our various conventions and inequality assumptions serve to show that

$$
A^2 \geq 0, \quad B^2 \geq 0, \quad C > 0, \quad \lambda^2 > 0, \quad \lambda AB \geq 0, \quad 0 \leq k^2 \leq 1.
$$

Were it instead true that $L^2 / I_2 < 2E \leq L^2 / I_1$, we would have $1 < k^2 \leq \infty$. This could be reduced to the normal $0 \leq k^2 < 1$ case by using the transformation formulae for elliptic functions, or by instead assigning the dn function to ω_1 and the cn function to ω_3.

If $k = 0$, then $2E = L^2 / I_3$. But then also $A^2 = B^2 = 0$, and the body rotates about the z axis in its minimum energy state. Excluding this degenerate case, since then the problem is trivially solvable, it follows that

$$
A^2 > 0, \quad B^2 > 0, \quad C > 0, \quad \lambda^2 > 0, \quad \lambda AB > 0, \quad 0 < k^2 \leq 1.
$$

The motion of the angular velocity in body coordinates is periodic. In terms of $K(k)$, the complete elliptic integral of the first kind, ω_1 and ω_2 have period $4K/\lambda$, and ω_3 has period $2K/\lambda$.

If k is close to 1, then K becomes large. Within each period, the body spends time $2K/\lambda$ rotating nearly about $+y$, its axis of middle inertia, and the other $2K/\lambda$ rotating nearly about $-y$. In a time of about $1/\lambda$ it flips around. Suppose the Earth behaved this way (which it doesn't because it is rotating about its axis of maximum inertia). Then an observer in the direction of Polaris would see polar bears on a counterclockwise rotating Earth. When the Earth flips, the observer would see penguins, also on a counterclockwise rotating Earth, because all the while, the angular momentum is of course unchanged.

Orientation

Although we now know the angular velocity of the body as a function of time, this is just a first integral; the complete solution of the problem requires in addition a determination of the orientation as a function of time.

The orientation of the body will be specified in terms of Euler angles. This representation has the virtue that each angle, as a function of time, is a constant rate of rotation plus a periodic libration (oscillation in angle). However, not all the periods are the same.

Let $OXYZ$ be a right-handed Cartesian coordinate system fixed in space, and let $Oxyz$ be a right-handed Cartesian coordinate system attached to the body and moving with it. Let ON lie along the intersection of planes OXY and Oxy and directed so that $ON = OZ \times Oz$. Let $Ox'y'z'$ be a coordinate system originally aligned with $OXYZ$. A sequence of rotations takes $Ox'y'z'$ into alignment with $Oxyz$.

First, rotate $Ox'y'z'$ about $Oz' = OZ$ by angle ϕ to bring Oy' into coincidence with ON.

Second, rotate $Ox'y'z'$ about $Oy' = ON$ by angle θ to bring Oz' into coincidence with Oz.

Third, rotate $Ox'y'z'$ about $Oz' = Oz$ by angle ψ to bring Ox' into coincidence with Ox as well as Oy' into coincidence with Oy.

The Euler angle representation is singular when z is parallel to Z, that is, when θ is 0 or π.

One can calculate the direction cosines relating the two sets of axes.

	X	Y	Z
x	$\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi$	$\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi$	$-\sin\theta\cos\psi$
y	$-\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi$	$-\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi$	$\sin\theta\sin\psi$
z	$\cos\phi\sin\theta$	$\sin\phi\sin\theta$	$\cos\theta$

Furthermore, the table above is also the rotation matrix R of section which operates on the fixed coordinates of a vector to give the body coordinates of the vector.

Choose the fixed Z axis to be along the constant angular momentum $\mathbf{L}$. Then in body coordinates

$$\begin{aligned} L_1 &= I_1\omega_1 = -L \sin\theta \cos\psi, \\ L_2 &= I_2\omega_2 = L \sin\theta \sin\psi, \\ L_3 &= I_3\omega_3 = L \cos\theta. \end{aligned}$$

Thus we now have

$$\begin{aligned} \sin\theta\cos\psi &= -\frac{I_1\omega_1}{L} = -A'\,\mathrm{cn}\,\lambda t, \quad A'^2 = \frac{I_1^2 A^2}{L^2} = \frac{I_1}{I_3 - I_1}\frac{2I_3E - L^2}{L^2}, \quad \mathrm{sign}\,A' = \mathrm{sign}\,A, \\ \sin\theta\sin\psi &= \frac{I_2\omega_2}{L} = B'\,\mathrm{sn}\,\lambda t, \quad B'^2 = \frac{I_2^2 B^2}{L^2} = \frac{I_2}{I_3 - I_2}\frac{2I_3E - L^2}{L^2}, \quad \mathrm{sign}\,B' = \mathrm{sign}\,B, \\ \cos\theta &= \frac{I_3\omega_3}{L} = C'\,\mathrm{dn}\,\lambda t, \quad C'^2 = \frac{I_3^2 C^2}{L^2} = \frac{I_3}{I_3 - I_1}\frac{L^2 - 2I_1E}{L^2}, \quad C' > 0. \end{aligned}$$

These equations suffice to determine θ and ψ as functions of time. From the bounds on the dn function, $k' \le \mathrm{dn}\,\lambda t \le 1$, and our assumption that $2E > L^2 / I_3$, we have

$$\begin{gathered} 0 < k'C' \le \cos\theta \le C' < 1, \\ 0 < \theta < \pi / 2. \end{gathered}$$

Because θ is bounded away from 0 and π, the singularities of the Euler angle representation are avoided. It also follows that θ can only librate; it cannot make complete rotations. The motion in θ is periodic with period $2K / \lambda$.

Next, we have, since $\sin\theta \ne 0$,

$$\tan\psi = -\frac{B'}{A'}\mathrm{sc}\,\lambda t, \quad \left(\frac{B'}{A'}\right)^2 = \frac{I_2}{I_1}\frac{I_3 - I_1}{I_3 - I_2} > 1, \quad \mathrm{sign}\,\lambda = \mathrm{sign}\,A'B'.$$

The sc function is tangent-like, being periodic and monotonically increasing (except across its poles) with zeroes at even multiples of K interleaving with poles at odd multiples of K. As λt increases from one multiple of K to the next, ψ *decreases* by π/2. Therefore ψ undergoes complete rotations, with rotational period $4K / \lambda$. However, the rotation rate, although always of the same sign, is not steady; a synchronous libration is superposed on the average motion. In other words,

$$\psi(t) = -\frac{\pi|\lambda|}{2K}t + f(t), \quad f(t + 2K / \lambda) = f(t).$$

At this point, we have the solution for two out of three of the Euler angles. In order to obtain an equation for the third Euler angle ϕ, refer to the rotation matrix R earlier in this section and the formula at the end of section by means of which the angular velocity components in body coordinates may be identified with elements of the antisymmetric matrix $(dR/dt)\,R^{-1}$. The calculation gives

$$\begin{aligned}
\omega_1 &= \frac{d\theta}{dt}\sin\psi - \frac{d\phi}{dt}\sin\theta\,\cos\psi,\\
\omega_2 &= \frac{d\theta}{dt}\cos\psi + \frac{d\phi}{dt}\sin\theta\,\sin\psi,\\
\omega_3 &= \frac{d\psi}{dt} + \frac{d\phi}{dt}\cos\theta.
\end{aligned}$$

Solve for the time derivatives of the Euler angles.

$$\begin{aligned}
\frac{d\psi}{dt} &= \omega_1\cot\theta\,\cos\psi - \omega_2\cot\theta\,\sin\psi + \omega_3,\\
\frac{d\theta}{dt} &= \omega_1\sin\psi + \omega_2\cos\psi,\\
\frac{d\phi}{dt} &= -\omega_1\csc\theta\,\cos\psi + \omega_2\csc\theta\,\sin\psi.
\end{aligned}$$

Substitute for the angular velocity body coordinates the values, found earlier, in terms of the projection of the angular momentum onto the body axes.

$$\begin{aligned}
\frac{d\psi}{dt} &= -L\cos\theta\left(\frac{\cos^2\psi}{I_1} + \frac{\sin^2\psi}{I_2} - \frac{1}{I_3}\right) < 0,\\
\frac{d\theta}{dt} &= -L\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin\theta\,\sin\psi\,\cos\psi,\\
\frac{d\phi}{dt} &= L\left(\frac{\cos^2\psi}{I_1} + \frac{\sin^2\psi}{I_2}\right) > 0.
\end{aligned}$$

The last of these three equations is the one of interest, and it can be written as

$$\frac{d\phi}{dt} = L\left(\frac{1}{I_1} - \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin^2\psi\right).$$

We have already shown that

$$\tan\psi = -\frac{B'}{A'}\operatorname{sc}\lambda t,\quad \left|\frac{B'}{A'}\right| > 1.$$

This inequality implies the existence of a real α such that

$$\left|\frac{B'}{A'}\right| = \sqrt{\frac{I_2}{I_1}\frac{I_3 - I_1}{I_3 - I_2}} = \operatorname{dn} i\alpha,\quad 0 < \alpha < K'.$$

Then

$$
\begin{aligned}
\sin^2\psi &= \frac{\tan^2\psi}{1+\tan^2\psi} = \frac{\mathrm{dn}^2\, i\alpha\, \mathrm{sc}^2\,\lambda t}{1+\mathrm{dn}^2\, i\alpha\, \mathrm{sc}^2\,\lambda t} \\
&= \frac{\mathrm{dn}^2\, i\alpha\, \mathrm{sn}^2\,\lambda t}{\mathrm{cn}^2\,\lambda t + \mathrm{dn}^2\, i\alpha\, \mathrm{sn}^2\,\lambda t} \\
&= \frac{\mathrm{dn}^2\, i\alpha\, \mathrm{sn}^2\,\lambda t}{1-\mathrm{sn}^2\,\lambda t + (1-k^2\,\mathrm{sn}^2\, i\alpha)\,\mathrm{sn}^2\,\lambda t} \\
&= \frac{\mathrm{dn}^2\, i\alpha\, \mathrm{sn}^2\,\lambda t}{1-k^2\,\mathrm{sn}^2\, i\alpha\, \mathrm{sn}^2\,\lambda t}.
\end{aligned}
$$

The zeroes of $\sin^2\psi$ are those of numerator, and the poles of $\sin^2\psi$ are the zeroes of the denominator, since the poles of the numerator cancel the poles of the denominator. The denominator vanishes when

$$
\mathrm{sn}\,\lambda t = \pm\frac{1}{k\,\mathrm{sn}\, i\alpha} = \pm\,\mathrm{sn}(i\alpha \pm iK')\,.
$$

So, modulo a period parallelogram $(2K, 2iK')$, $\sin^2\psi$ has simple poles at

$$
\lambda t = z_\pm = \pm i(K'-\alpha)\,,
$$

and a double zero at 0.

As an elliptic function, $\sin^2\psi$ is determined, up to an additive constant, by the principal part at its poles. Let $r_\pm$ be the residue at $z_\pm$. Since $\mathrm{dn}\, i\alpha$ is real, $\sin^2\psi$ is a real function, and the residues $r_\pm$ at the complex conjugate points $z_\pm$ must be complex conjugates. Also, the sum of the residues in a period parallelogram must be 0. Therefore $r_\pm$ must be pure imaginary: $r_\pm = \pm ir$.

The function

$$
g(z) = \frac{\vartheta_1'(z,\tau)}{\vartheta_1(z,\tau)} = \frac{d}{dz}\log\vartheta_1(z,\tau)
$$

is quasi-periodic, satisfying

$$
g(z+\pi) = g(z), \quad g(z+\pi\tau) = g(z) - 2i\,,
$$

and has simple poles at $m\pi + n\pi\tau$, for integer *m* and *n*, with residue 1 at each pole. If $f(z)$ is an elliptic function having periods P_1 and $P_2 = \tau P_1$, and only simple poles, and these located at $\gamma_1,\ldots,\gamma_n$ with corresponding residues $r_1,\ldots,r_n$, then

$$
f(z) = C + \frac{\pi}{P_1}\sum_{1\le k\le n} r_k\, g\!\left(\frac{\pi(z-\gamma_k)}{P_1}\right),
$$

for some constant *C*. This theorem generalizes to higher order poles, but for us, it shows that

$$\sin^2\psi = \frac{\pi i r}{2K}\left(g\left(\frac{\pi(z-z_+)}{2K}\right) - g\left(\frac{\pi(z-z_-)}{2K}\right) + C\right), \quad z=\lambda t, \quad \tau = \frac{iK'}{K},$$

where C is independent of z. Thus

$$\sin^2\psi = \frac{\pi i r}{2K}\left(\frac{\vartheta_1'\left(\frac{\pi(\lambda t - iK' + i\alpha)}{2K}\right)}{\vartheta_1\left(\frac{\pi(\lambda t - iK' + i\alpha)}{2K}\right)} - \frac{\vartheta_1'\left(\frac{\pi(\lambda t + iK' - i\alpha)}{2K}\right)}{\vartheta_1\left(\frac{\pi(\lambda t + iK' - i\alpha)}{2K}\right)} + C\right)$$

$$= \frac{\pi i r}{2K}\left(\frac{\vartheta_4'\left(\frac{\pi(\lambda t + i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t + i\alpha)}{2K}\right)} - \frac{\vartheta_4'\left(\frac{\pi(\lambda t - i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t - i\alpha)}{2K}\right)} + C\right).$$

When $\lambda t = 0$, $\operatorname{sn}\lambda t = 0$, and $\sin^2\psi = 0$. This determines C, so that now we have

$$\sin^2\psi = \frac{\pi i r}{2K}\left(\frac{\vartheta_4'\left(\frac{\pi(\lambda t + i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t + i\alpha)}{2K}\right)} - \frac{\vartheta_4'\left(\frac{\pi(\lambda t - i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t - i\alpha)}{2K}\right)} - 2\frac{\vartheta_4'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_4\left(\frac{\pi i\alpha}{2K}\right)}\right).$$

When $\lambda t = iK'$, $\operatorname{sn}\lambda t = \infty$, and

$$\sin^2\psi = \frac{\operatorname{dn}^2 i\alpha}{-k^2\operatorname{sn}^2 i\alpha} = \frac{\operatorname{dn}^2 i\alpha}{\operatorname{dn}^2 i\alpha - 1} = \frac{I_2}{I_3}\frac{I_3 - I_1}{I_2 - I_1},$$

while the right-hand side becomes

$$\frac{\pi i r}{K}\left(\frac{\vartheta_1'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_1\left(\frac{\pi i\alpha}{2K}\right)} - \frac{\vartheta_4'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_4\left(\frac{\pi i\alpha}{2K}\right)}\right)$$

This last expression can be simplified to Jacobi elliptic functions. If the sn function is expressed in terms of theta functions:

$$\operatorname{sn} z = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2} z)}{\vartheta_4(\vartheta_3^{-2} z)}, \quad \vartheta_3^{-2} = \frac{\pi}{2K},$$

we have

$$\frac{d}{dz}\log \operatorname{sn} z = \vartheta_3^{-2}\left(\frac{\vartheta_1'(\vartheta_3^{-2}z)}{\vartheta_1(\vartheta_3^{-2}z)} - \frac{\vartheta_4'(\vartheta_3^{-2}z)}{\vartheta_4(\vartheta_3^{-2}z)}\right)$$

$$= \frac{\pi}{2K}\left(\frac{\vartheta_1'\left(\frac{\pi z}{2K}\right)}{\vartheta_1\left(\frac{\pi z}{2K}\right)} - \frac{\vartheta_4'\left(\frac{\pi z}{2K}\right)}{\vartheta_4\left(\frac{\pi z}{2K}\right)}\right)$$

On the other hand,

$$\frac{d}{dz}\log \operatorname{sn} z = \frac{\operatorname{cn} z \operatorname{dn} z}{\operatorname{sn} z}.$$

Therefore

$$\frac{\vartheta_1'\left(\frac{\pi z}{2K}\right)}{\vartheta_1\left(\frac{\pi z}{2K}\right)} - \frac{\vartheta_4'\left(\frac{\pi z}{2K}\right)}{\vartheta_4\left(\frac{\pi z}{2K}\right)} = \frac{2K}{\pi}\frac{\operatorname{cn} z \operatorname{dn} z}{\operatorname{sn} z},$$

and the condition determining r becomes

$$\frac{I_2}{I_3}\frac{I_3 - I_1}{I_2 - I_1} = 2ir\frac{\operatorname{cn} i\alpha \operatorname{dn} i\alpha}{\operatorname{sn} i\alpha}.$$

Since $0 < \alpha < K'$, each of $\operatorname{dn} i\alpha$, $\operatorname{cn} i\alpha$, and $-i \operatorname{sn} i\alpha$ is positive. Then

$$\operatorname{dn} i\alpha = \sqrt{\frac{I_2}{I_1}\frac{I_3 - I_1}{I_3 - I_2}},$$

$$-i \operatorname{sn} i\alpha = \frac{1}{k}\sqrt{\operatorname{dn}^2 i\alpha - 1} = \sqrt{\frac{I_3}{I_1}\frac{L^2 - 2I_1E}{2I_3E - L^2}},$$

$$\operatorname{cn} i\alpha = \sqrt{1 - \operatorname{sn}^2 i\alpha} = \sqrt{\frac{I_3 - I_1}{I_1}\frac{L^2}{2I_3E - L^2}}.$$

Solving for r gives

$$r = \frac{1}{2(I_2 - I_1)L}\sqrt{\frac{I_1 I_2 (I_3 - I_2)(L^2 - 2I_1E)}{I_3}}$$

$$= \frac{I_1 I_2 |\lambda|}{2(I_2 - I_1)L}.$$

We can now express the right-hand side of the equation for $d\phi / dt$ in the desired form.

$$\frac{d\phi}{dt}=\frac{L}{I_1}+\frac{\pi|\lambda|}{4iK}\left(\frac{\vartheta_4'\left(\frac{\pi(\lambda t+i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t+i\alpha)}{2K}\right)}-\frac{\vartheta_4'\left(\frac{\pi(\lambda t-i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t-i\alpha)}{2K}\right)}-2\frac{\vartheta_4'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_4\left(\frac{\pi i\alpha}{2K}\right)}\right),$$

which is ready for integration. Collect together the terms independent of t by defining

$$\mu=\frac{L}{I_1}-\frac{\pi|\lambda|}{2iK}\frac{\vartheta_4'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_4\left(\frac{\pi i\alpha}{2K}\right)}.$$

Then

$$\begin{aligned}\frac{d\phi}{dt}&=\mu+\frac{\pi|\lambda|}{4iK}\left(\frac{\vartheta_4'\left(\frac{\pi(\lambda t+i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t+i\alpha)}{2K}\right)}-\frac{\vartheta_4'\left(\frac{\pi(\lambda t-i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(\lambda t-i\alpha)}{2K}\right)}\right)\\&=\mu+\frac{1}{2i}\frac{d}{dt}\log\left(\frac{\vartheta_4\left(\frac{\pi(|\lambda|t+i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(|\lambda|t-i\alpha)}{2K}\right)}\right),\end{aligned}.$$

the last step being justified because ϑ_4 is an even function. Therefore

$$\phi=\mu t+\frac{1}{2i}\log\left(\frac{\vartheta_4\left(\frac{\pi(|\lambda|t+i\alpha)}{2K}\right)}{\vartheta_4\left(\frac{\pi(|\lambda|t-i\alpha)}{2K}\right)}\right)=\mu t+\arg\vartheta_4\left(\frac{\pi(|\lambda|t+i\alpha)}{2K}\right).$$

The first term on the right shows that ϕ rotates at an average rate of μ, or an average rotational period of $2\pi/\mu$. The second term is a superposed periodic libration with period $2K/\lambda$.

The state of motion of the rigid body is completely specified by the three coordinates ψ, θ, ϕ and their velocities $d\psi/dt$, $d\theta/dt$, $d\phi/dt$. For motion with fixed angular momentum, we have chosen a coordinate system such that the velocities become functions of ψ and θ. If in addition the energy is fixed, then θ, which is between 0 and $\pi/2$, becomes a function of ψ since

$$\frac{2E}{L^2}=\frac{\sin^2\theta\cos^2\psi}{I_1}+\frac{\sin^2\theta\sin^2\psi}{I_2}+\frac{\cos^2\theta}{I_3}.$$

Thus for a given rigid body with given initial conditions, the motion is confined to a two dimensional manifold, in fact a torus, with coordinates ψ and ϕ. The ψ coordinate, modulo 2π, is a periodic function of time with period $4K/\lambda$, and since θ and the three velocities are uniquely determined by ψ, these also have the same period. Actually, these last four variables have fundamental period $2K/\lambda$. However, during the time $4K/\lambda$ in which ψ undergoes one rotation, ϕ, in addition to undergoing exactly two libration cycles, rotates by the amount $4K\mu/\lambda$. Since in general, this angle is not a rational multiple of 2π, the motion of the rigid body is "almost periodic" and the path in phase space is a non-closed curve which threads around the torus with an irrational pitch, never intersecting itself, and densely filling the torus. Since the velocities are independent of ϕ, the curve is congruent to itself under a rotation in ϕ by an integral multiple of $4K\mu/\lambda$. Of course, since the ratio $(2K/\pi)(\mu/\lambda)$ is a continuous function of the physical parameters, there exists a dense set (of measure zero) of these parameters for which the ratio is rational and the motion is periodic with a suitably long period, in the sense of mathematical perfection which ignores the inevitable perturbations to which real physical systems are subject.

Example

If $2E = L^2/I_2$, then $k = 1$, and K becomes infinite. This is the interesting special case mentioned earlier, where the angular velocity path passes through the middle inertia axis.

In this limit, sn(z) = tanh(z), cn(z) = sech(z), and dn(z) = sech(z). Let $\omega_0 = L/I_2$. Then

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{I_2}{I_1}\frac{I_3 - I_2}{I_3 - I_1}}\ \omega_0 \operatorname{sech}\lambda t, \\
\omega_2 &= \omega_0 \tanh\lambda t, \\
\omega_3 &= \sqrt{\frac{I_2}{I_3}\frac{I_2 - I_1}{I_3 - I_1}}\ \omega_0 \operatorname{sech}\lambda t, \\
\lambda &= \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}\ \omega_0.
\end{aligned}$$

For all time except one brief flip episode, of duration on the order of $1/\lambda$, the body rotates about its axis of middle inertia with angular velocity $\pm\omega_0$. The ratio ω_0/λ is the number of rotations during a flip. From section , $I_3 - I_2 \le I_1$, and therefore $\omega_0^2/\lambda^2 \ge I_3/(I_2 - I_1) > 1$, so that a flip can never be faster than a rotation.

Continuing on to the Euler angles, we have, since sc(z) =sinh(z),

$$\begin{aligned}
\theta &= \arccos(C'\operatorname{sech}\lambda t), \quad C' = \sqrt{\frac{I_3}{I_2}\frac{I_2 - I_1}{I_3 - I_1}} < 1, \\
\psi &= -\arctan\left(\frac{B'}{A'}\sinh\lambda t\right), \quad \frac{B'}{A'} = \sqrt{\frac{I_2}{I_1}\frac{I_3 - I_1}{I_3 - I_2}} > 1.
\end{aligned}$$

Thus during the flip, ψ changes from $\pi/2$ to $-\pi/2$, while θ, which normally resides at $\pi/2$, makes a brief excursion.

To obtain the motion of ϕ, note that in addition to $K \to \infty$, we also have $K' = \pi / 2$, $\tau = iK' / K \to 0$, and $\tau' = iK / K' \to i\infty$.

By the Jacobi imaginary transformation,

$$\vartheta_4(z,\tau) = \frac{1}{\sqrt{-i\tau}} \exp\left(\frac{i\tau' z^2}{\pi}\right) \vartheta_2(\tau' z, \tau'),$$

$$\frac{\vartheta_4'(z,\tau)}{\vartheta_4(z,\tau)} = \frac{2i\tau' z}{\pi} + \tau' \frac{\vartheta_2'(\tau' z,\tau')}{\vartheta_2(\tau' z,\tau')}.$$

Also, since $q' = \exp(i\pi\tau') \to 0$, we have

$$\vartheta_2(z,\tau') = 2q'^{1/4} \cos z, \quad \frac{\vartheta_2'(z,\tau')}{\vartheta_2(z,\tau')} = -\tan z\,.$$

For calculating the value of μ, we use

$$z = \frac{\pi i\alpha}{2K}, \quad \tau' z = -\alpha\,,$$

so that

$$\frac{\pi}{2iK} \frac{\vartheta_4'\left(\frac{\pi i\alpha}{2K}\right)}{\vartheta_4\left(\frac{\pi i\alpha}{2K}\right)} = \frac{1}{\tau'}\left(\tau' \tan\alpha - \frac{2i\alpha}{\pi}\right) \to \tan\alpha\,.$$

Since $0 < \alpha < K' = \pi / 2$,

$$\sqrt{\frac{I_2}{I_1}\frac{I_3 - I_1}{I_3 - I_2}} = \mathrm{dn}\, i\alpha = \sec\alpha, \quad \tan\alpha = \sqrt{\frac{I_3}{I_1}\frac{I_2 - I_1}{I_3 - I_2}}\,,$$

and then

$$\mu = \frac{L}{I_1} - \lambda\tan\alpha = \frac{L}{I_2} = \omega_0\,.$$

Since θ and ψ remain constant except during the flip episode, this constant rate of rotation in ϕ is to be expected.

For the ϕ libration term, use, in the same Jacobi imaginary transformation,

$$z = \frac{\pi(\lambda t + i\alpha)}{2K}, \quad \tau' z = -\alpha + i\lambda t, \quad \frac{i\tau' z^2}{\pi} = -\frac{(\lambda t + i\alpha)^2}{2K} \to 0\,.$$

Then

$$\begin{aligned}\arg\vartheta_4(z,\tau) &= \arg\vartheta_2(\tau' z,\tau') \\ &= \arg\cos(-\alpha + i\lambda t) \\ &= \arctan(\tan\alpha\tanh\lambda t),\end{aligned}$$

where the arctan is in the range $-\pi/2$ to $+\pi/2$.

Thus during the flip, ϕ speeds up a bit so as to gain a rotational phase angle of 2α.

References

1. E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th edition, Cambridge University Press, 1937. The rigid body problem is solved in section 69.
2. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, 1927. Chapter XX covers the general theory of elliptic functions and the Weierstrass functions. Chapter XXI covers theta functions. Chapter XXII covers the Jacobi elliptic functions and elliptic integrals.