

Strange Series for Sierpinski's Gasket

Bill Gosper
Julian Ziegler Hunts

Abstract

We derive the Fourier coefficients using unusual matrix products and integer sequences. From these we get a direct enumeration of points along the curve, along with the Fourier series describing the subset of $\mathbb{C}$ representable in base 2 using the digits $\{0, 1, e^{\pm 2i\pi/3}\}$. Two equivalent recursive definitions of a “gasketfilling curve” lead to different formulas for the Fourier coefficients whose equivalence we have been unable to prove.

The gasket set $\mathbf{G}$

We shall take $\mathbf{G}$ to be the closure of any of the bounded sets satisfying

$$2S = (S + 1) \cup (S + \omega) \cup (S + \omega^2),$$

where adding a complex number z to a set means add z to all of its elements, and ω is the complex cube root of 1:

$$\omega = e^{2i\pi/3}, \quad \omega^2 = \omega^{-1}.$$

We note incidentally that some S , *e.g.* the infinite union of line segments suggested by the figure

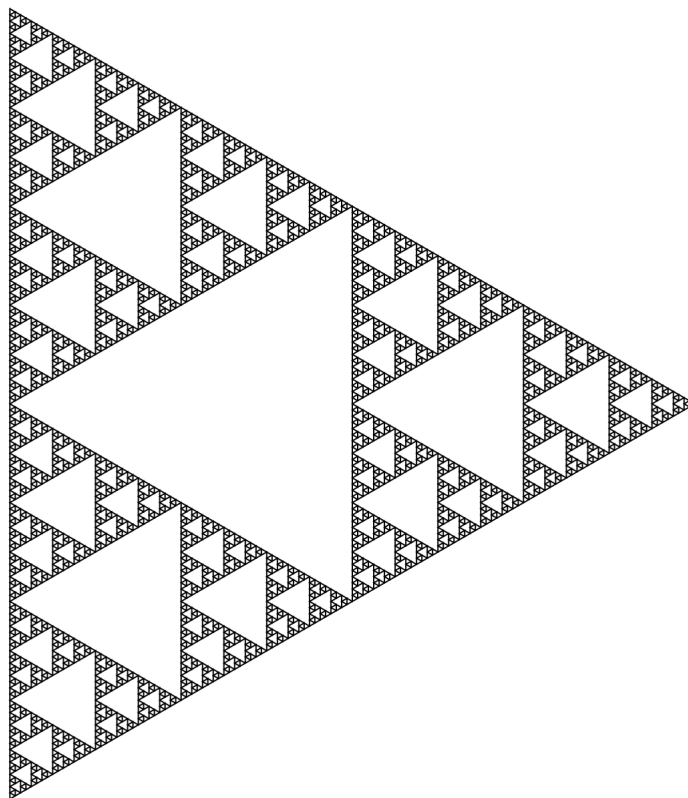


Figure 1: A small subset of $\mathbf{G}$

are very much smaller than $\mathbf{G}$, just as Cantor's (uncountable) subset of the unit interval is the closure of the very much smaller (countable) set of endpoints of the deleted thirds. For example the point $z = \frac{5+\sqrt{3}i}{14} \notin S$ because neither $\Re(z)$, $\Re(\omega)$, nor $\Re(\omega^2)$ is a dyadic rational, which would be necessary to lie on one of the segments. Yet $z \in \mathbf{G}$ because it satisfies $2z = 1 + \omega z$ while $2\mathbf{G} = (\mathbf{G} + 1) \cup \dots$ and $\mathbf{G} = \omega\mathbf{G}$ (by rotational symmetry). A direct construction of $\mathbf{G}$ is to start with the closed triangle and delete successively smaller open triangles:

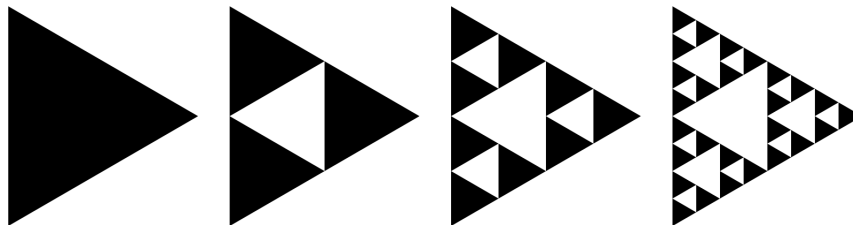


Figure 2: Constructing $\mathbf{G}$ by attrition

To reemphasize: This process leaves very much more than the boundaries of the triangles (the line segments in Figure 1). In fact, Figure 2 could also depict an open triangle suffering the deletion of closed triangles, leaving no boundary points at all, and nothing to draw by way of conventional depictions of points and line segments, which are very misleading because you can see them! Any set with less than two dimensions has no area and would be invisible, except implicitly as the boundary between visible sets.

Graphically experimenting with the gasket may bolster one's intuition. Inverting it in a concentric circle is at least pretty:

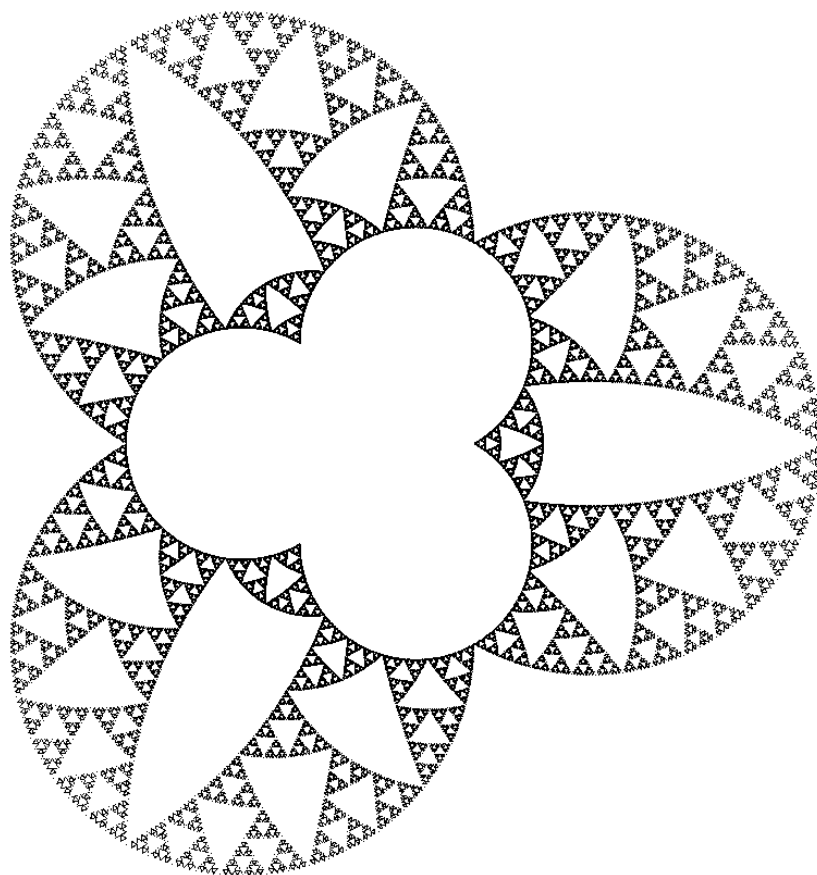


Figure 3: $\mathbf{G}$ inverted in a concentric circle

More interesting is inverting in a circle centered on a vertex (in this case, the rightward pointing one):

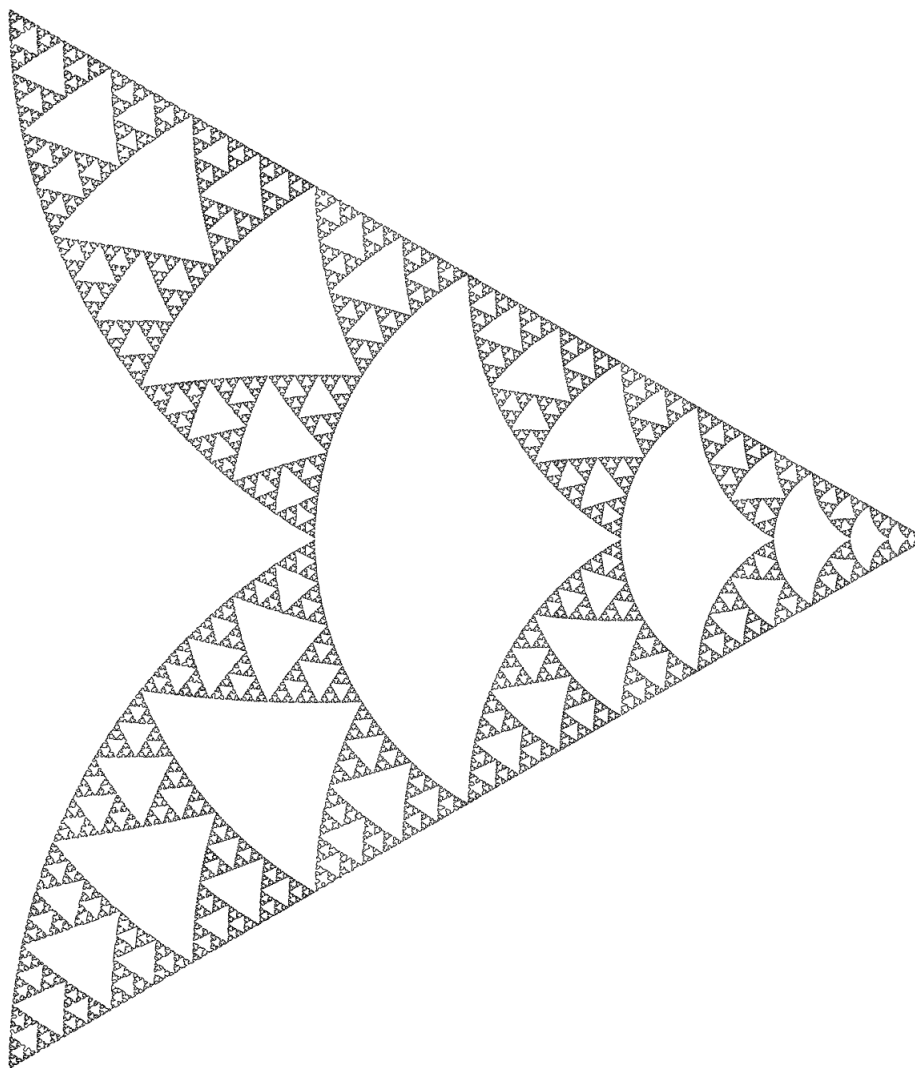


Figure 4: $\mathbf{G}$ inverted in a circle centered at the rightmost vertex

This picture is a bit subtle. What appears to be a vertex on the right is really a greatly reduced image of the gasket's left edge, which takes the form of a tiny 60° arc. The flaring extends infinitely leftward, showing ever greater magnification of the neighborhood of the vertex formerly at the origin.

The curve $G(t)$

A continuous map $G([-1, 1])$ onto G is suggested by the ternary recursive scheme

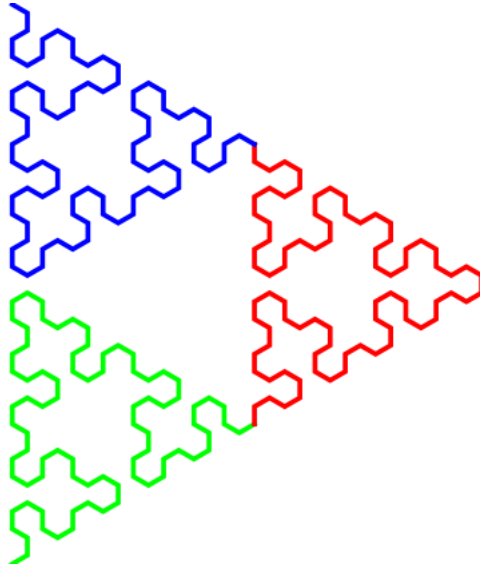


Figure 5: A recursive construction of $G(t)$

which is in fact an accurate, connect-the-dots polygon sampling the actual curve at equal intervals. If we suppose the curve joins $-i$ to i (instead of ω^2 to ω) as $-1 \leq t \leq 1$, we can use either of two equivalent recursive definitions:

$$G(t) = \frac{1}{2} \begin{cases} \omega^2(G(-3t-2) - \sqrt{3}) & -1 \leq t \leq -1/3 \\ G(3t) + \sqrt{3} & -1/3 \leq t \leq 1/3 \\ \omega(G(2-3t) - \sqrt{3}) & 1/3 \leq t \leq 1 \end{cases}$$

or

$$G(t) = \frac{1}{2} \begin{cases} \omega^2(\overline{G(3t+2)} - \sqrt{3}) & -1 \leq t \leq -1/3 \\ G(3t) + \sqrt{3} & -1/3 \leq t \leq 1/3 \\ \omega(\overline{G(3t-2)} - \sqrt{3}) & 1/3 \leq t \leq 1 \end{cases}.$$

Note that, for rational t , we don't need termination conditions, *e.g.* to evaluate $G(-5/6)$,

$$G\left(-\frac{5}{6}\right) = \frac{1}{2}\omega^2\left(G\left(\frac{1}{2}\right) - \sqrt{3}\right) = \frac{1}{4}\left(G\left(\frac{1}{2}\right) - 3i\right)$$

so

$$G\left(\frac{1}{2}\right) = \frac{2\sqrt{3}+3i}{7} \text{ and } G\left(-\frac{5}{6}\right) = \frac{\sqrt{3}-9i}{14}$$

Alternatively,

$$G\left(-\frac{5}{6}\right) = \frac{1}{2}\omega^2\left(\overline{G\left(-\frac{1}{2}\right)} - \sqrt{3}\right) = \frac{1}{4}\left(G\left(\frac{1}{2}\right) - 3i\right) = \frac{1}{8}\omega^2\left(2\sqrt{3}+3i + \overline{G\left(-\frac{1}{2}\right)}\right).$$

So

$$\overline{G\left(-\frac{1}{2}\right)} = \frac{2\sqrt{3}+3i}{7} \text{ and } G\left(-\frac{5}{6}\right) = \frac{\sqrt{3}-9i}{14}.$$

In like manner (or two),

$$G\left(-\frac{3}{7}\right) = \frac{\sqrt{3}-2i}{3}, G\left(\frac{1}{13}\right) = \frac{16\sqrt{3}+i}{21}, G\left(\frac{7}{22}\right) = \frac{530\sqrt{3}+429i}{1023}, G\left(\frac{14}{39}\right) = \frac{3\sqrt{3}+3i}{7}, G\left(\frac{23}{40}\right) = \frac{52\sqrt{3}+69i}{255}$$

Since a continuous function is fully specified by its values at rational points, this recursion suffices to define $G(t)$.

G's Fourier series

We note that the three cases $-1 \leq t \leq -1/3$, $-1/3 \leq t \leq 1/3$, and $1/3 \leq t \leq 1$ in the recursion correspond to the first digit of t being -2, 0, and 2 respectively, when t is written in base three with digits $0, \pm 2$ (which we shall call *even* balanced ternary). For ease of manipulation, we try to write this recursion in matrix form. Unfortunately, in two of the three cases, the G value is conjugated, so we need to write out $G(t) = g_1(t) + g_2(t)i$, $g_1(t), g_2(t) \in \mathbb{R}$ for our manipulations. This leads to $\begin{pmatrix} g_1(t) \\ g_2(t) \\ 1 \end{pmatrix} = M_d \begin{pmatrix} g_1(3t-d) \\ g_2(3t-d) \\ 1 \end{pmatrix}$, where $d = 2(\lfloor \frac{3t+3}{2} \rfloor - 1)$ is the first digit of t in (even) balanced ternary, and

$$M_{-2} = \begin{pmatrix} \frac{-1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 \end{pmatrix}, M_0 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} \frac{-1}{4} & \frac{-\sqrt{3}}{4} & \frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose that $t = 0.d_1d_2d_3d_4d_5 \dots$ in (even) balanced ternary, and let $t_1 = 0.d_2d_3d_4d_5 \dots$, $t_2 = 0.d_3d_4d_5d_6 \dots$, etc. Then we have that $\begin{pmatrix} g_1(t) \\ g_2(t) \\ 1 \end{pmatrix} = M_{d_1} \begin{pmatrix} g_1(t_1) \\ g_2(t_1) \\ 1 \end{pmatrix} = M_{d_1}M_{d_2} \begin{pmatrix} g_1(t_2) \\ g_2(t_2) \\ 1 \end{pmatrix} = \left(\prod_{i=1}^k M_{d_i} \right) \begin{pmatrix} g_1(t_k) \\ g_2(t_k) \\ 1 \end{pmatrix}$, and taking the limit as $k \rightarrow \infty$ (which we can do, since the first two elements of the column vector become decreasingly important), we have that $\begin{pmatrix} g_1(t) \\ g_2(t) \\ 1 \end{pmatrix} = \left(\prod_{i=1}^{\infty} M_{d_i} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which gives $G(t) = \begin{pmatrix} 1 & i & 0 \end{pmatrix} \left(\prod_{i=1}^{\infty} M_{d_i} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Now we let $\chi = e^{-i\pi x}$, and consider $\begin{pmatrix} 1 & i & 0 \end{pmatrix} \left(\prod_{n=1}^{\infty} \frac{\chi^{-2 \cdot 3^{-n}} M_{-2} + M_0 + \chi^{2 \cdot 3^{-n}} M_2}{3} \right) \begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}$. The first two partial products (replacing the ∞ by 1 and 2) are

$$\begin{pmatrix} 1 & i & 0 \end{pmatrix} \left(\frac{\chi^{-2/3} M_{-2} + M_0 + \chi^{2/3} M_2}{3} \right) \begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix} = \frac{\chi^{-2/3} G(-2/3) + G(0) + \chi^{2/3} G(2/3)}{3}$$

and

$$\begin{aligned} & \begin{pmatrix} 1 & i & 0 \end{pmatrix} \left(\frac{\chi^{-2/3} M_{-2} + M_0 + \chi^{2/3} M_2}{3} \right) \left(\frac{\chi^{-2/9} M_{-2} + M_0 + \chi^{2/9} M_2}{3} \right) \begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{9} \left(\chi^{-8/9} G(-8/9) + \chi^{-2/3} G(-2/3) + \chi^{-4/9} G(-4/9) + \chi^{-2/9} G(-2/9) + G(0) \right. \\ & \quad \left. + \chi^{2/9} G(2/9) + \chi^{4/9} G(4/9) + \chi^{2/3} G(2/3) + \chi^{8/9} G(8/9) \right). \end{aligned}$$

The partial products will continue to be Riemann sums, so the limit is

$$A(x) := \begin{pmatrix} 1 & i & 0 \end{pmatrix} \left(\prod_{n=1}^{\infty} \frac{\chi^{-2 \cdot 3^{-n}} M_{-2} + M_0 + \chi^{2 \cdot 3^{-n}} M_2}{3} \right) \begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \int_{-1}^1 \chi^t G(t) dt = \frac{1}{2} \int_{-1}^1 e^{-i\pi x t} G(t) dt.$$

Since the partial products of the matrices (without the row and column vector) are powers of χ times products of M_d s, so the first two entries in the $\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}$ are unimportant (this could also be shown by noting that the first two entries in the bottom row are 0, so that the upper-left 2×2 can be computed without reference to the rest of the

matrix, and it has norm (using the sum of absolute values matrix norm) less than $19/20$, so the limit of the upper-left 2×2 is the zero matrix). Using deMoivre's theorem, we compute

$$\mathcal{M}_n := \frac{\chi^{-2 \cdot 3^{-n}} M_{-2} + M_0 + \omega^{2 \cdot 3^{-n}} M_2}{3} = \begin{pmatrix} \frac{1}{6} \left(1 - \cos \left(\frac{2\pi x}{3^n} \right) \right) & \frac{i}{2\sqrt{3}} \sin \left(\frac{2\pi x}{3^n} \right) & \frac{1}{2\sqrt{3}} \left(1 + \cos \left(\frac{2\pi x}{3^n} \right) \right) \\ \frac{i}{2\sqrt{3}} \sin \left(\frac{2\pi x}{3^n} \right) & \frac{1}{6} \left(1 + \cos \left(\frac{2\pi x}{3^n} \right) \right) & \frac{-i}{2} \sin \left(\frac{2\pi x}{3^n} \right) \\ 0 & 0 & \frac{1}{3} \left(1 + 2 \cos \left(\frac{2\pi x}{3^n} \right) \right) \end{pmatrix}.$$

Thus, we have (assuming that the integral exists, which is reasonable and confirmed numerically) that,

$$G(t) = \sum_{k=-\infty}^{\infty} A(k) e^{i\pi k t}, \quad A(x) = \begin{pmatrix} 1 & i & 0 \end{pmatrix} \prod_{n=1}^{\infty} \mathcal{M}_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

for all $-1 < t < 1$. The sum $G(t)$ converges very slowly, halving the error of approximation for each trebling of the number of harmonics. But the $A(x)$ product converges rapidly, like 9^{-n} .

Since a Fourier series interpolates the midpoint of a jump discontinuity,

$$-i = G(-1) \neq \sum_{k=-\infty}^{\infty} A(k) e^{-i\pi k} = 0 = \sum_{k=-\infty}^{\infty} A(k) e^{i\pi k} \neq G(1) = i.$$

Summing $-69 \leq k \leq 69$ and plotting $-1 < t < 1$, we see the jump, along with some distortion of the endpoints of the curve, due to Gibbs ringing, another inherent property of Fourier-approximated discontinuities.

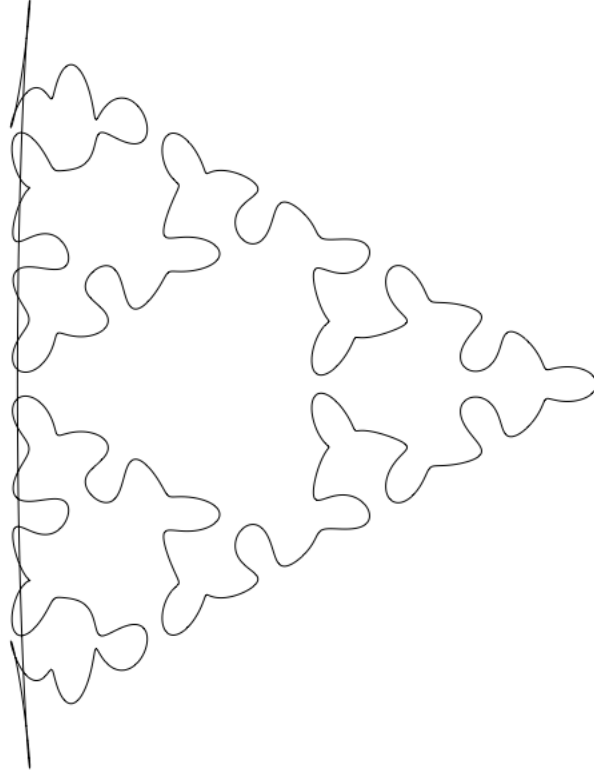


Figure 6: Discontinuously closing the loop causes Gibbs ringing.

Due to the gasket's self-similarity, we can remove this particular discontinuity by linking three copies of $G(t)$ in a triangular loop:

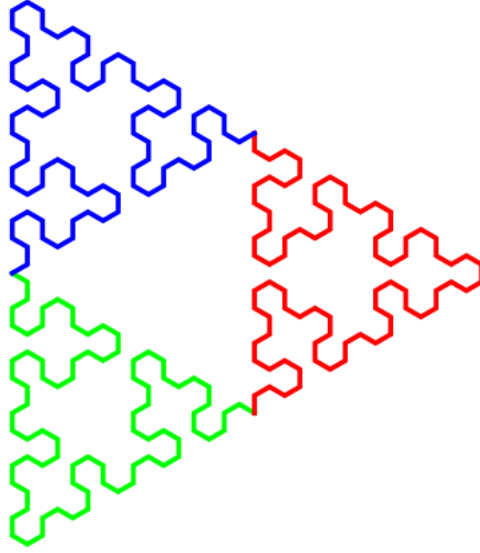


Figure 7: Sweeping the set $\mathbf{G}$ with a closed curve

But what is its Fourier series? The following is lifted almost verbatim from *Lisp + Calculus = Identities*, in *Artificial Intelligence and Mathematical Theory of Computation*, Papers in Honor of John McCarthy, (Vladimir Lifschitz, ed., 1991).

Fourier rotational symmetry

Let $z(t)$, $-1 \leq t \leq 1$, be an arc in the complex plane (such as $G(t)$), and let $z_m(t)$ be the period $2m$ arrangement of m such arcs around a regular m -gon, so that $z_m(t+2) = e^{2i\pi/m} z_m(t)$. Suppose that z_m is nice enough to equal its Fourier approximation,

$$z_m(t) = \sum_j a_j e^{i\pi j t/m}.$$

Then the Fourier coefficients are:

$$\begin{aligned} a_j &= \frac{1}{2m} \int_{-m}^m e^{-i\pi j t/m} z_m(t) dt = \frac{1}{2m} \int_{-1}^{2m-1} e^{-i\pi j t/m} z_m(t) dt \\ &= \frac{1}{2m} \sum_{n=0}^{m-1} \int_{-1}^1 e^{-i\pi j(t+2n)/m} z_m(t+2n) dt \\ &= \frac{1}{2m} \sum_{n=0}^{m-1} e^{-i\pi(j-1)2n/m} \int_{-1}^1 e^{-i\pi j t/m} z_m(t) dt \\ &= \begin{cases} \frac{1}{2} \int_{-1}^1 e^{-i\pi j t/m} z_m(t) dt & \text{if } m|j-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Let us further stipulate that the arcs remain congruent as we vary m , and each of the $z_m(t)$ is rotated so as to differ from $z(t)$ by only a constant $c_m := c \cot(\pi/m) = c \cot(\pi(k+1/m))$, for $-1 \leq t \leq 1$ (for fixed side length, m -gons are proportional to $\cot(\pi/m)$) (note that we are implicitly assuming that $z(-1) = -z(1)$, otherwise we would have

$c_m = c \cot(\pi/m) + c'$ for some c'). Then we can write

$$\begin{aligned} z_m(t) &= \sum_k a_{km+1} e^{i\pi(k+1/m)t} = \sum_k A^+(k+1/m) e^{i\pi(k+1/m)t}, \\ A^+(x) &:= \frac{1}{2} \int_{-1}^1 e^{-i\pi x t} (z(t) + c \cot(\pi x)) dt \\ &=: A(x) + c \frac{\cos \pi x}{\pi x}. \end{aligned}$$

Thus, if we can compute the function $A(x) = \frac{1}{2} \int_{-1}^1 e^{-i\pi x t} z(t) dt$, which we have already done for $z(t) = G(t)$, we know the Fourier series that arranges the arcs around an arbitrary m -gon. This is actually rather magic, and is not equivalent to taking the m th root of some $z_1(t)$, which would distort the arcs, or to taking every m th term of some series free of m . In fact m can be fractional, which arranges $\text{numerator}(m)$ arcs around a “star” (some of the m s in the calculation above would need to be replaced by $\text{numerator}(m)$, but the end result is the same), or negative, in which case the arcs are oriented in the opposite direction (pointing in instead of pointing out, in the case of G , which has the shape of a triangle pointing rightward and has $c = 1$). For the closed-loop gasket sweep, we need $m = 3$. Truncating the Fourier sum to $|k| \leq 50$ or to $|k| \leq 105$, and plotting $-3 \leq t \leq 3$, we see a “low pass” (temporal, not spatial) filtration of the (unrectifiable) gasket sweep.

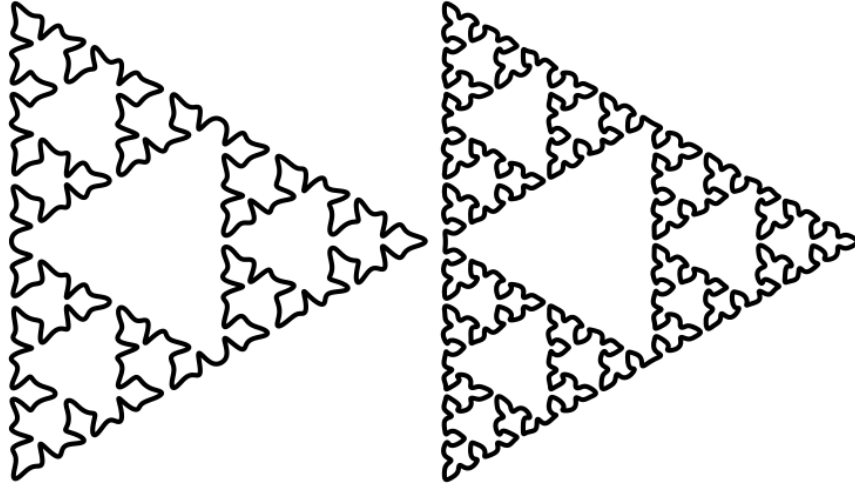


Figure 8: Approximations (through harmonics $\pm 50 + \frac{1}{3}$ and $\pm 105 + \frac{1}{3}$) to the closed loop gasket sweep

Plotting instead $m = -6$, harmonics through ± 239 , $-6 \leq t \leq 6$ produces this doily:

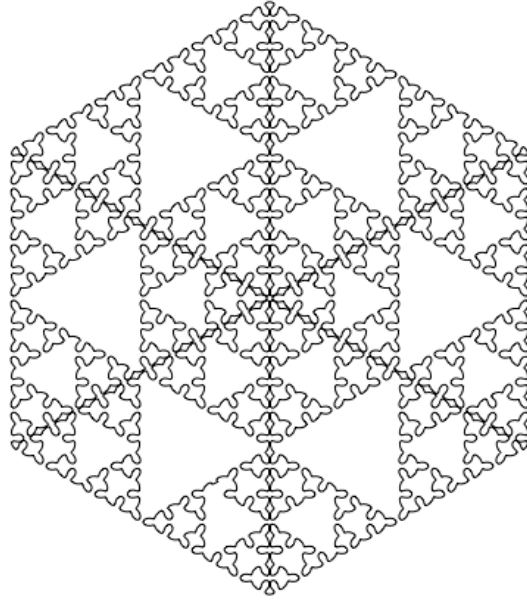


Figure 9: A Fourier doily

Choosing instead $m = -3$, harmonics through ± 105 , $-3 \leq t \leq 3$ sweeps the gasket thrice, in different orders, and with various winding numbers around the interior triangular voids:

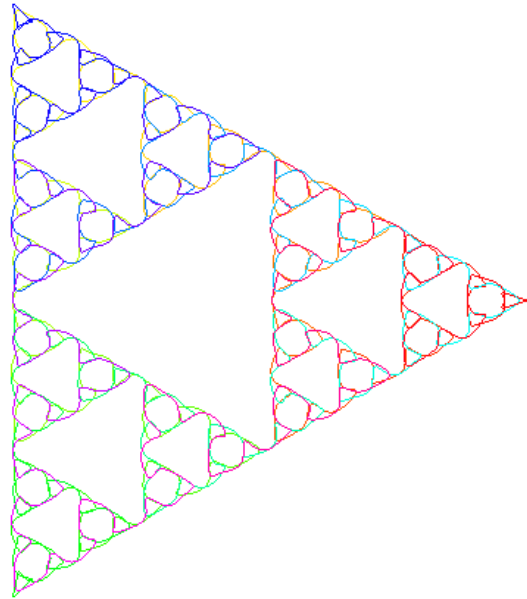


Figure 10: $\mathbf{G}$ traced three ways by one function

For $m = 2$ we'd clearly get the rhombus formed by two gaskets edge to edge. For $m = 1$, instead of the Gibbs ringing Figure 6, we divide by zero due to the $k = -1, x = k + 1/1$ case of $(\cos \pi x)/\pi x$ term, which is absent from $A(x)$. Discarding just that term creates a continuous loop which is no longer a gasket.

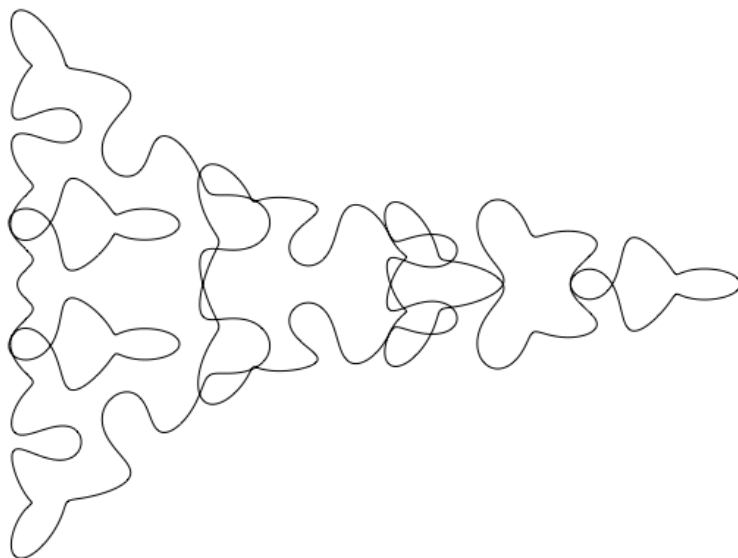


Figure 11: Gasket trying to fit on a “1 gon”.

But now, if we simply add the linear (non-Fourier) term it , we get an infinite vertical strip of perfect gaskets.

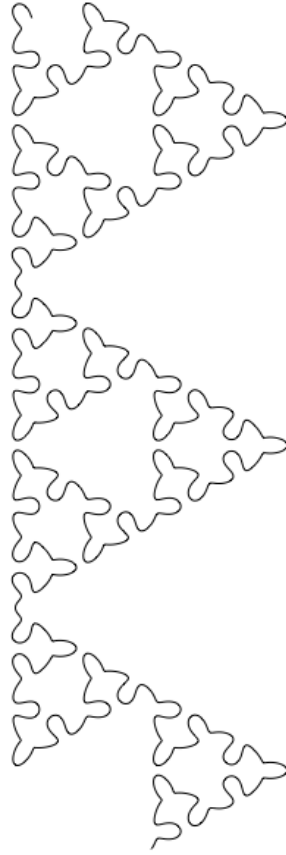


Figure 12: Unrolled gaskets forever.

If only we had some way to alternately point the six gaskets of Figure 9 inward and outward, we would have the boundary of the (two dimensional) “base 2 fractal” satisfying

$$2S = S \cup (S + 1) \cup (S + \omega) \cup (S + \omega^2),$$

part of which Mandelbrot called the Arrowhead fractal:

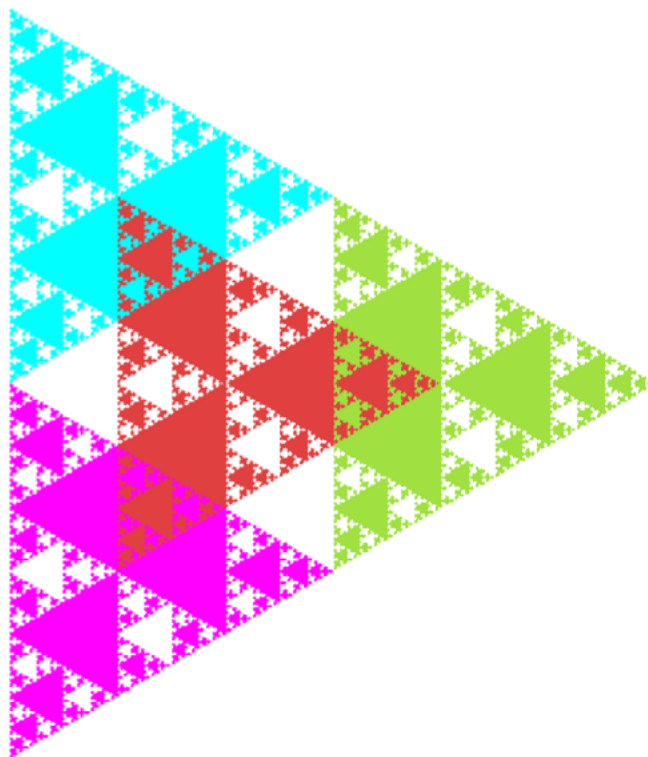


Figure 13: Red = $0\dots$, green = $1\dots$, cyan = $\omega\dots$, magenta = $\bar{\omega}\dots$

This figure accommodates a closed-loop spacefill (which visits every point at least twice, and countably many six times):

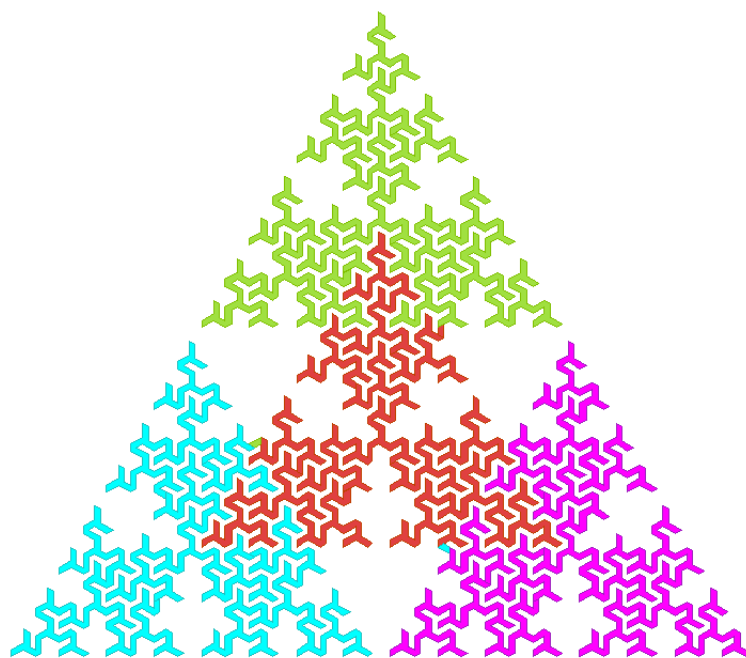


Figure 14: Self-complementary polygonal sampling of base 2 spacefill

Six copies of this figure pack improbably around a point one third the way along a side:



Figure 15: Serious meshing

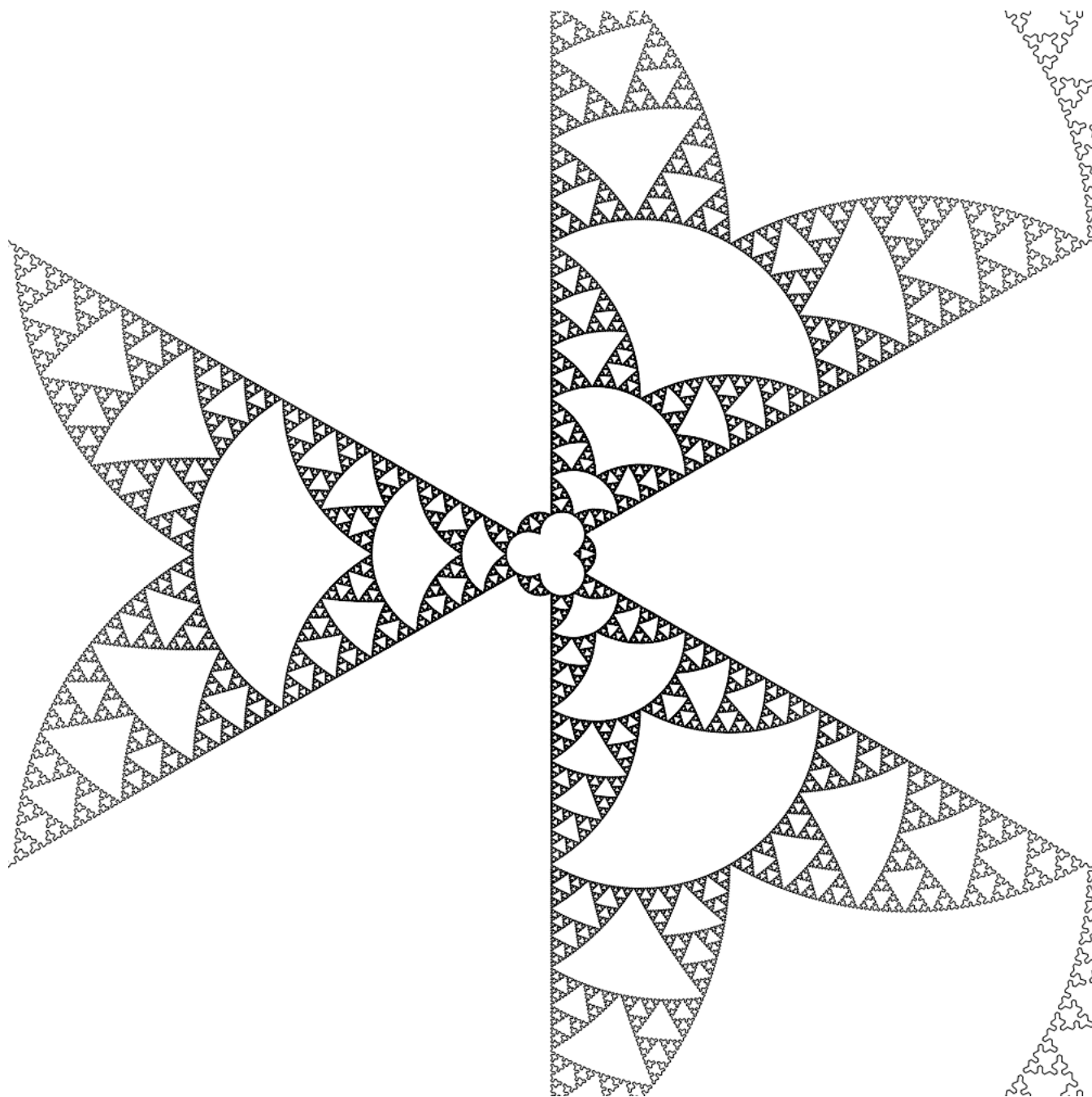


Figure 16: Polygonal approximation to a closed-loop sweep of the base 2 boundary, inverted in a circle

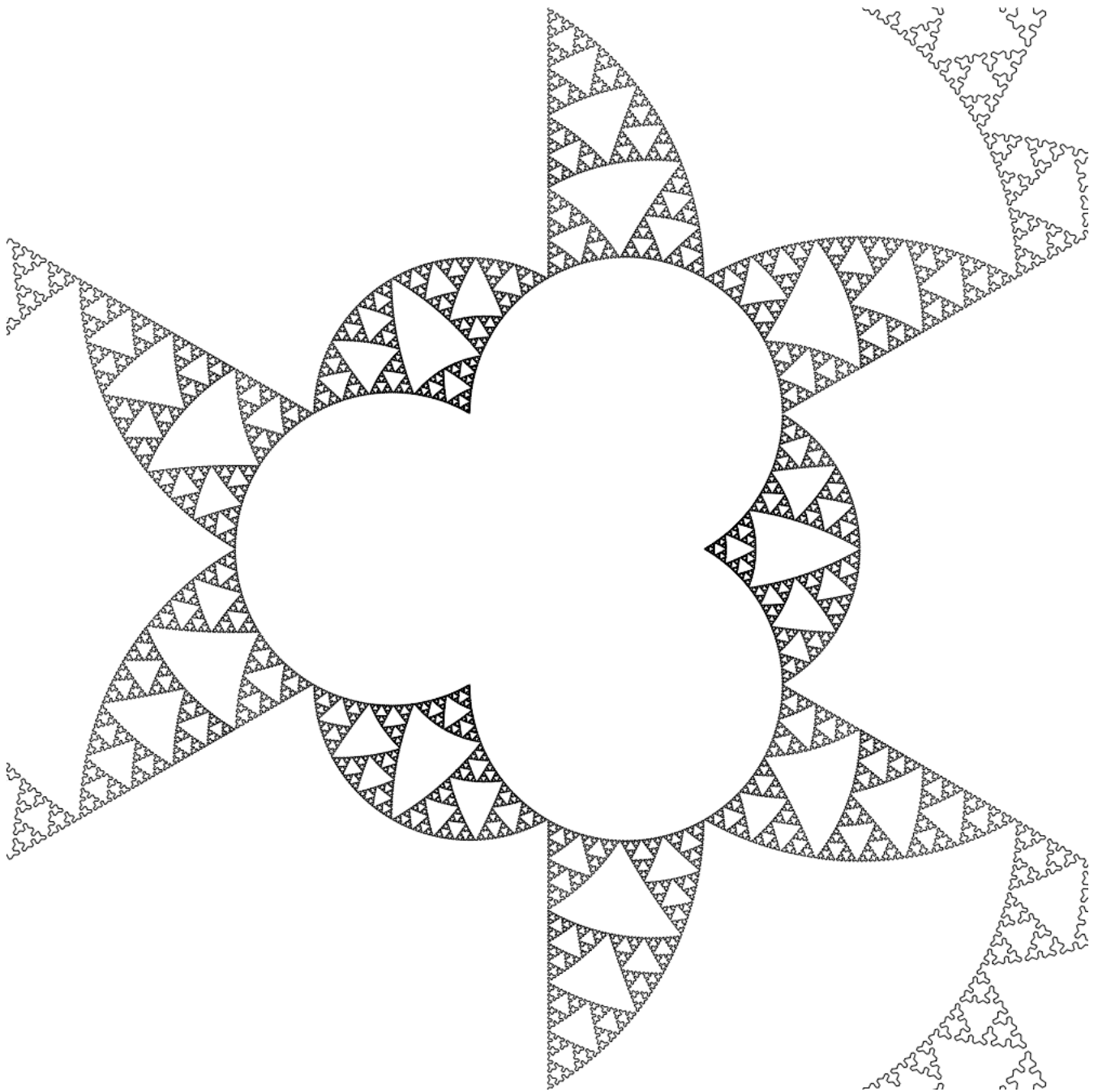


Figure 17: Closeup of the inversion center

Generalizing our analysis of rotational symmetry, we can in fact derive formulas to arrange arcs alternately pointing inward and outward around a $2m$ -gon (where the original arc $z(t)$ goes from $-i$ to i , and pointing inward means the appropriate translation and rotation of $-\overline{z(t)}$ (the other possibility is $-z(-t)$, and they're equivalent for $z = G$)),

you have the Fourier series

$$\begin{aligned}
z_m^{\text{alt}}(t) &= \sum_k b_k e^{i\pi(k/2+1/(2m))t}, & b_k &:= \frac{1}{4} \int_{-1}^1 e^{-i(k/2+1/(2m))\pi t} (z(t) - (-1)^k \overline{z(t)}) dt + \frac{\cos(k\pi/2) \cos(\pi/(2m))}{k\pi/2 + \pi/(2m)} \\
&= \frac{A(k/2 + 1/(2m)) - (-1)^k \overline{A(-k/2 - 1/(2m))}}{2} + \frac{2 \cos(k\pi/2) \cos(\pi/(2m))}{\pi(k + 1/m)} \\
&= \begin{cases} \frac{1}{2} \int_{-1}^1 e^{-i(k+1/m)\pi t/2} \Im(z(t)) dt + \frac{2}{\pi} \frac{(-1)^{k/2} \cos(\pi/(2m))}{k+1/m} & k \text{ even} \\ \frac{1}{2} \int_{-1}^1 e^{-i(k+1/m)\pi t/2} \Re(z(t)) dt & k \text{ odd} \end{cases},
\end{aligned}$$

where $A(x) := \frac{1}{2} \int_{-1}^1 e^{-i\pi x t} z(t) dt$ as before. (For “inward” meaning $-z(-t)$, replace $\overline{z(t)}$ by $z(-t)$ in the first formula for a_k , or $\overline{A(x)}$ by $A(x)$ in the second.) Choosing $m = 3$, summing the 139 lowest harmonics, and plotting for $-3 < t < 3$:

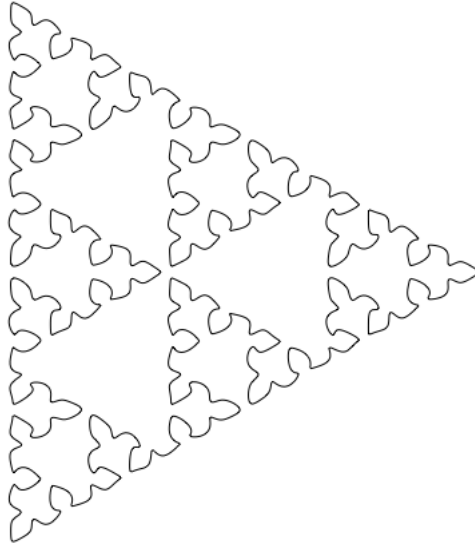


Figure 18: Fourier-approximated boundary of the base 2 complex fractions

About when the junior author was celebrating his 0th birthday, the senior author found this much niftier 2×2 product for $A^+(x)$:

$$\begin{pmatrix} A^+(x) & -A^+(x) \\ -A^+(-x) & A^+(-x) \end{pmatrix} = \frac{2}{\pi x} \prod_{n=1}^{\infty} \begin{pmatrix} \frac{1}{2} & -\sin\left(\frac{\pi}{6} + \frac{2\pi x}{3^n}\right) \\ -\sin\left(\frac{\pi}{6} - \frac{2\pi x}{3^n}\right) & \frac{1}{2} \end{pmatrix},$$

which converges “only” like 3^{-n} . Unfortunately, the senior author has become so senior that he can’t recall how he did this, requiring the junior author to derive the aforementioned 3×3 product for $A(x)$. Old emails also claim to express, for similarly forgotten reasons, $A^+(x)$ as the limit of a sum:

$$A^+(x) = \frac{1}{\pi x} \lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{k=-\lfloor 3^n/2 \rfloor}^{\lfloor 3^n/2 \rfloor} (-1)^k \cos\left(2\pi \left(\frac{d(k)}{3} + \frac{kx}{3^n}\right)\right),$$

where $d(k)$ is a peculiar three-valued function described next, after we point out that this limit is of a sum of 3^n terms, but divided by 2^n .

The function $d(k)$

There are at least three equivalent definitions, based on partitioning all the integers into six disjoint sets (a, b, c, A, B, C) (Alternatively, (Bob, Carol, Ted, Alice, Herb, and Sanchez).) The sets are

$$\begin{aligned}
 a &= \dots, -40, -36, -32, -30, -28, -16, -12, -10, -4, 0, 4, 10, 12, 16, 28, 30, 32, 36, 40, \dots & (2 * A019989) \\
 b &= \dots, -34, -20, 2, 6, 8, 14, 18, 22, 24, 26, 38, \dots & (2 * A019990) \\
 c &= \dots, -38, -26, -24, -22, -18, -14, -8, -6, -2, 20, 34, \dots & (2 * A019991) \\
 A &= \dots, -25, -21, -19, -7, 7, 19, 21, 25, \dots \\
 B &= \dots, -39, -37, -31, -27, -23, -17, -151 - 13, -9, -5, -3, -1, 11, 29, 33, 35, \dots \\
 C &= \dots, -35, -33, -29, -11, 1, 3, 5, 9, 13, 15, 17, 23, 27, 31, 37, 39, \dots,
 \end{aligned}$$

where the $A0199\dots$ are the lookup keys in Sloane's On-Line Encyclopedia of Integer Sequences. These six sets can be rapidly generated by initializing (a, b, c, A, B, C) to $(\{\}, \{\}, \{\}, \{\}, \{\}, \{\})$ and then iterating

$$\begin{aligned}
 (a, b, c, A, B, C) \leftarrow & (\{0\} \cup 3a \cup 3C + 1 \cup 3B - 1, \\
 & 3b \cup 3C - 1 \cup 3A + 1, \\
 & 3c \cup 3B + 1 \cup 3A - 1, \\
 & 3A \cup 3c - 1 \cup 3b + 1, \\
 & 3B \cup 3c + 1 \cup 3a - 1, \\
 & 3C \cup 3a + 1 \cup 3b - 1),
 \end{aligned}$$

which expands as follows.

$$\begin{aligned}
 & (\{0\}, \{\}, \{\}, \{\}, \{\}, \{\}) \\
 & (\{0\}, \{\}, \{\}, \{\}, \{-1\}, \{1\}) \\
 & (\{-4, 0, 4\}, \{2\}, \{-2\}, \{\}, \{-3, -1\}, \{1, 3\}) \\
 & (\{-12, -10, -4, 0, 4, 10, 12\}, \{2, 6, 8\}, \{-8, -6, -2\}, \{-7, 7\}, \{-13, -9, -5, -3, -1, 11\}, \{-11, 1, 3, 5, 9, 13\}) \\
 & (\{-40, -36, -32, -30, -28, -16, -12, -10, -4, 0, 4, 10, 12, 16, 28, 30, 32, 36, 40\}, \\
 & \quad \{-34, -20, 2, 6, 8, 14, 18, 22, 24, 26, 38\}, \\
 & \quad \{-38, -26, -24, -22, -18, -14, -8, -6, -2, 20, 34\}, \\
 & \quad \{-25, -21, -19, -7, 7, 19, 21, 25\}, \\
 & \quad \{-39, -37, -31, -27, -23, -17, -15, -13, -9, -5, -3, -1, 11, 29, 33, 35\}, \\
 & \quad \{-35, -33, -29, -11, 1, 3, 5, 9, 13, 15, 17, 23, 27, 31, 37, 39\}).
 \end{aligned}$$

The unions are all magically disjoint.

The function d classifies the even integers $a \cup b \cup c$: $d(a) \bmod 3 = 0, d(b) \bmod 3 = 2, d(c) \bmod 3 = 1$. The predicates for membership in each of the six sets are mutually recursive:

$$a?(0) := \text{True}, \quad b?(0) := c?(0) := A?(0) := B?(0) := C?(0) := \text{False}$$

$$\frac{n}{3} \quad \text{or} \quad \frac{n-1}{3} \quad \text{or} \quad \frac{n+1}{3}, \text{ whichever is an integer}$$

$$\begin{aligned}
 a?(n) &:= a?() \text{ or } C?() \text{ or } B?(), \\
 b?(n) &:= b?() \text{ or } A?() \text{ or } C?(), \\
 c?(n) &:= c?() \text{ or } B?() \text{ or } A?(), \\
 A?(n) &:= A?() \text{ or } b?() \text{ or } c?(), \\
 B?(n) &:= B?() \text{ or } c?() \text{ or } a?(), \\
 C?(n) &:= C?() \text{ or } a?() \text{ or } b?().
 \end{aligned}$$

There is a direct but rather strange formula for $d(k)$: Write k in (ordinary) balanced ternary. Delete the 0 digits. Then form the alternating sum of the remaining digits. *E.g.*, for $k = -69$, the balanced ternary is $\bar{1}0110$. Losing the 0s: $\bar{1}11$. Alternating sum: $\bar{1} - 1 + 1 = -1$. Some values near 0:

k	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	...
$d(k)$	...	-1	0	-1	-2	-1	0	1	2	1	0	1	2	3	2	1	...
$d(k) \bmod 3$	...	$\bar{1}$	0	$\bar{1}$	1	$\bar{1}$	0	1	$\bar{1}$	1	0	1	$\bar{1}$	0	$\bar{1}$	1	...

$\bar{1} := -1$.

The sequence $d(k) \bmod 3$ is “square free” (stutter free). Another surprise: if we replace $A^+(x)$ by the limit(sum) form in the explicit Fourier series, it is possible to interchange the two summations, with the result that the running sum of

$$\left(-\frac{1}{2}\right)^{|-1+(d(2k) \bmod 3)|} + \left(-\frac{1}{2}\right)^{|d(2k+2)|} e^{i\pi/3}$$

for $\lfloor -729/4 \rfloor \leq k \leq \lfloor 729/4 \rfloor$, say, draws a gasket!

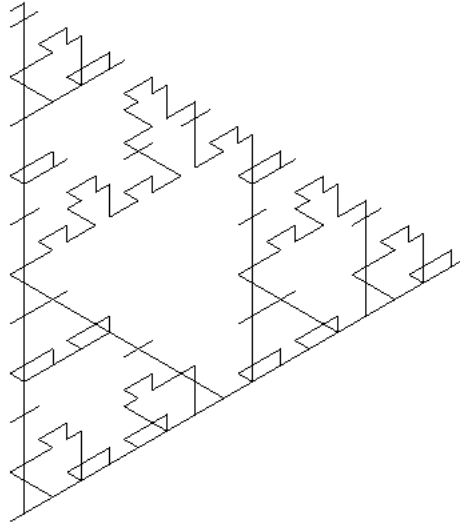


Figure 19: $d(k)$ draws the gasket directly.

A chance discovery (based on an erroneous formula) leads to an even simpler formula for a sequence of points along G . “Connecting the dots” of consecutive partial sums of

$$\sum_{k=-\lfloor 3^7/4 \rfloor}^p \omega^{d(2k)}, \quad -\lfloor 3^7/4 \rfloor \leq p \leq \lfloor 3^7/4 \rfloor$$

produces the curve

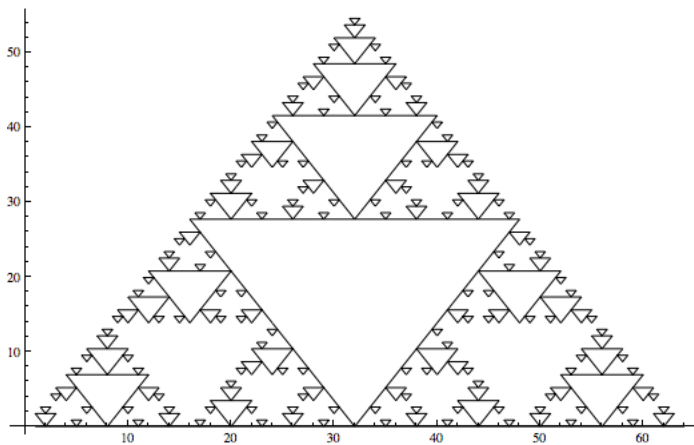


Figure 20: $d(2k)$ draws the gasket even more directly.

Using the odd terms instead,

$$\sum_{k=-\lfloor 3^7/4 \rfloor}^p \omega^{d(2k+1)}, \quad \lfloor 1/2 - 3^7/4 \rfloor \leq p \leq \lfloor 3^7/4 - 1/2 \rfloor$$

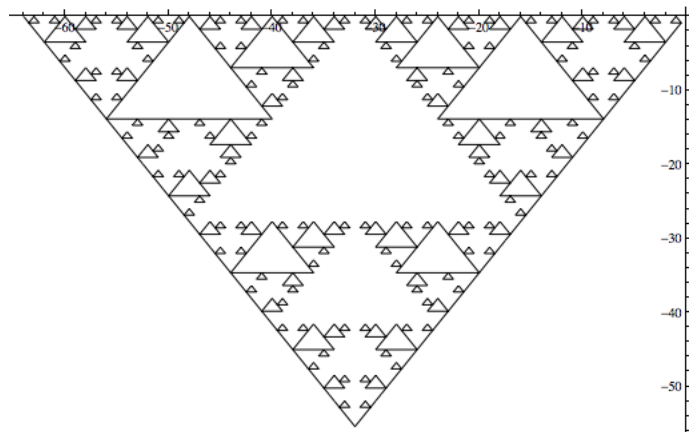


Figure 21: $d(2k + 1)$ works too.