Stirlingoid approximations to three factorialoids, with exact figures of merit

Legend:

Out[248]= $$\prod_{n}^{\infty} \frac{\texttt{approximation}[n]}{n!} == \texttt{figureofmerit}$$

The traditional Stirling's,

In[249]:= $$\# == \texttt{Activate@\#}\ \&@\prod_{n}^{\infty} \frac{e^{-n}\, n^{n} \sqrt{2\,\pi\, n}}{n!}$$

Out[249]= $$\prod_{n}^{\infty} \frac{e^{-n}\, n^{n+\frac{1}{2}} \sqrt{2\,\pi}}{n!} == 0$$

is meritless. The approximation is so poor that the Product of all the relative errors diverges. But replacing $\sqrt{n}$ by $\sqrt{n + 1/6}$,

In[255]:= $$\texttt{Append}[\#, \texttt{N@\#[[2]]}]\ \&\left[\# == \texttt{Activate@\#}\ \&@\prod_{n}^{\infty} \frac{e^{-n}\, n^{n} \sqrt{2\,\pi\,(n+1/6)}}{n!}\ \texttt{// nogamma}\right]$$

Out[255]= $$\prod_{n}^{\infty} \frac{e^{-n}\, n^{n} \sqrt{n+\frac{1}{6}}\ \sqrt{2\,\pi}}{n!} == \frac{A^{2}}{\sqrt[12]{e}\ \sqrt[4]{2\,\pi}\ \sqrt{\frac{1}{6}!}} == 0.992249561947291$$

A merit of 1.0 is perfect. *A* is "Glaisher's constant",

In[258]:= `Append[#, N@#[[2]]] &[# == deGlaisher@# &@Glaisher]`

Out[258]= $$A == e^{\frac{1}{12}-\zeta^{(1,0)}(-1,1)} == 1.2824271291005334`$$

based on the derivative of $\zeta(s) = \zeta(s,1)$ at $s = -1$.

Expanding the Productand at ∞ gives a more direct estimate of quality:

In[259]:= $$\texttt{Series}\left[\frac{e^{-n}\, n^{n} \sqrt{2\,\pi\, n}}{n!}, \{n, \infty, 1\}\right]$$

Out[259]= $$1 - \frac{1}{12\, n} + O\left[\frac{1}{n}\right]^{2}$$

(which is why the Product diverged.)

In[260]:= $$\texttt{Series}\left[\frac{e^{-n}\, n^{n} \sqrt{2\,\pi\,(n+1/6)}}{n!}, \{n, \infty, 2\}\right]$$

Out[260]= $$1 - \frac{1}{144\, n^{2}} + O\left[\frac{1}{n}\right]^{3}$$

Invoking a third order approximation,

In[∘]:=
```
Series[(E^-n Sqrt[n] (1/(12 n) + n)^n Sqrt[2 π])/n!, {n, ∞, 3}]
```

Out[∘]= $1 - \frac{1}{1440\, n^3} + O\left[\frac{1}{n}\right]^4$

In[263]:=
```
Append[#, N@#[[2]] // Chop] &[
 Product[(E^-n Sqrt[n] (1/(12 n) + n)^n Sqrt[2 π])/n!, {n, ∞}] ==
  (Glaisher^2 ((I/(2 Sqrt[3]))!/(-(I/(2 Sqrt[3])))!)^(I/(2 Sqrt[3])))/
   (2^(1/4) E^(1/6) π^(1/4) Abs[Hyperfactorial[I/(2 Sqrt[3])]]^2)]
```

Out[263]= $$\prod_{n}^{\infty} \frac{e^{-n} \sqrt{n}\left(n+\frac{1}{12 n}\right)^{n} \sqrt{2 \pi}}{n!} == \frac{\left(\frac{\frac{i}{2\sqrt{3}}!}{\left(-\frac{i}{2\sqrt{3}}\right)!}\right)^{\frac{i}{2\sqrt{3}}} A^2}{\sqrt[4]{2}\ \sqrt[6]{e}\ \sqrt[4]{\pi}\ \left|H\left(\frac{i}{2\sqrt{3}}\right)\right|^2} == 0.998823164760701$$

(Exercise: why is this r.h.s. "obviously" real?)

Finally, here is a "Stirling's" for Hyperfactorial(n) := $1^1 \times 2^2 \times 3^3 \ldots n^n$:

In[264]:=
```
Append[#, N@#[[2]] // Chop] &[Product[(E^(-k^2/4) Glaisher k^(1/12 + k/2 + k^2/2))/Hyperfactorial[k], {k, ∞}] == (E^((3 Zeta[3])/(8 π^2)) (2 π)^(1/24))/Sqrt[Glaisher]]
```

Out[264]= $$\prod_{k}^{\infty} \frac{A\, e^{-\frac{k^2}{4}}\, k^{\frac{k^2}{2}+\frac{k}{2}+\frac{1}{12}}}{H(k)} == \frac{\sqrt[24]{2\pi}\ e^{\frac{3\zeta(3)}{8\pi^2}}}{\sqrt{A}} == 0.99787591723441$$

In[267]:=
```
Series[(E^(-k^2/4) Glaisher k^(1/12 + k/2 + k^2/2))/Hyperfactorial[k], {k, ∞, 3}] // Simplify
```

Out[267]= $1 - \frac{1}{720\, k^2} + O\left[\frac{1}{k}\right]^{13/6}$